OL Academy

ITCS254
DISCRETE STRUCTURES I

T. Maryam Al-Shahab

Chapter1
1.1 Propositional Logic

Definition (1)

A proposition is a declarative sentence that is either true or false.
In another words it is a sentence that states a true or false fact and a sentence without a fact giving a fact are not considered as proposition.

Examples:

1. Washington, D.C., is the capital of the United States of America. True Proposition

2. Toronto is the capital of Canada. False Proposition

3. 1+1 = 2. True Proposition

4. 2+2 = 3. False Proposition

5. What time is it? Not a Proposition

6. Read this carefully. Not a Proposition

7. x+1 = 2. Not a Proposition

8. x+y = z. Not a Proposition

Definition (2)

Assuming that **p** is a proposition the negation of **p** will be denoted by **¬p** which is represented in the sentence: “it is not the case that”.

Examples:

1. p = “Michael’s PC runs Linux”
Negation: “It is not the case that Michael’s PC runs Linux”.

Simplified negation: “Michael’s PC does not run Linux”

¬p = “Michael’s PC does not run Linux”.

2. **q = “Jassim’s smartphone has at least 64GB of memory”**
Negation: It is not the case that Jassim’s smartphone has at least 64GB of memory.

Simplified negation: “Jassim’s smartphone does not have at leat 64GB of memory,

or more simple: **q = “Jassim’s smartphone has more than 64GB if memory”.**

3. **r = “At least 10 inches of rain fell today in Manama”**
Negation: **¬r = “Less than 10 inches of rain fell today in Manama.**

Definition (3)

Let p and q be propositions the conjunction of p and q is denoted by **p ∧ q** is the proposition p and q.

Important note:
The conjunction rule is p ∧ q is true when both p and q are true otherwise the statement is false.

Definition (4)

Let p and q be proposition the disjunction of p and q is denoted by **p ∨ q** is the proposition p or q.

Important note:
The disjunction rule is when both p and q are false then the statement is false else if p or q one of them is true then the statement is true.

Definition (5)

Let p and q be proposition the exclusive or of p and q is denoted by $\mathbf{p \oplus q}$.

Important note:
The rule is the statement is true when exactly one of p and q is true otherwise the statement is false.

Conditional statements:

Definition (6)

Let p and q be proposition the conditional statement $\mathbf{p \rightarrow q}$ is the proposition of "if p then q".

Important note:
The conditional statement rule if p is true and q is false the statement is false otherwise the statement is true.

In the conditional statement p is called the hypothesis or the premise and q is called the conclusion or the consequence.

The different ways to express $p \rightarrow q$

"if p, then q"	"p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q is necessary for p"
"q unless ¬p"	

Exercise

Write each of these statements in the form “if p, then q” in English:

1. **It snows whenever the wind blows from the northeast.**
   if the wind blows from the northeast then it snows.

2. **The apples trees will born bloom if it stays warm for a week.**
   if it stays warm for a week then the apple trees will born bloom.

3. **That the Pistons win the championsip implies that they beat the Lakers.**
   if that the Pistons win the championsip then they beat the Lakers.

4. **If you drive than 400 miles, you will need to buy gasoline.**
   if you drive than 400 miles, then you will need to buy gasoline.

5. **Your guarantee is good only if you bought your CD player less than 90 days ago.**
   if your guarantee is good then you bought your CD player less than 90 days ago.

6. **Jan will go swimming unless the water is too cold.**

Given in the question q unless ¬p and we want the form is if p then q

First step: we need to get p:

¬p= The water is too cold.
p = The water is not too cold.

Second step:

if that water is not too cold, then jan will go swimming.

Definition (7)

Converse of $\mathbf{p \rightarrow q}$ **is** $\mathbf{q \rightarrow p}$

Contrapositive of $\mathbf{p \rightarrow q}$ **is** $\mathbf{\neg q \rightarrow \neg p}$

Inverse of $\mathbf{p \rightarrow q}$ **is** $\mathbf{\neg p \rightarrow \neg q}$

Definition (8)

Let p and q be propositions the biconditionals statement $\mathbf{q} \leftrightarrow \mathbf{p}$ is the proposition up “if and only if".

More ways to represent a biconditionals statement:

"p is necessary and sufficient for q".

"if p then q, and conversely".

"p iff q".

Truth table of compound proposition:

p	q	$\neg p$	$\neg q$	$p \vee q$	$p \wedge q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \oplus q$
T	T	F	F	T	T	T	T	F
T	F	F	T	T	F	F	F	T
F	T	T	F	T	F	T	F	T
F	F	T	T	F	F	T	T	F

$\neg p$: Nagation of p.

$\neg q$: Nagation of q.

$p \vee q$: if there is **T** either in p or q the statement is **T**.

$p \wedge q$: p and q should both be **T** for the statement to be **T**.

$p \rightarrow q$: if p is **F** the statement directly is **T** but if p is **T** then q should be **T** for the statement to **T**.

$p \leftrightarrow q$: p and q both should be the same for the statement to be **T**.

$p \oplus q$: p and q should be different for the statement to be **T**.

Precedence of logic operators:

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Translating English sentences:

1. Consider the sentences: "It is below freezing and snowing", " It is below freezing but not snowing". Let consider the propositional variables:
p = "It is below freezing", and q = "It is raining" so we can wright: $p \wedge q$, $p \wedge \neg q$

It is below freezing and it is raining, it is freezing and it is not raining.

2. How can this English sentence be translated into a logical expression? "You can access the Internet from campus only if you are computer science major or you are not a freshman". we lat a,c, and f represent "You can access the Internet from campus", " you are computer science major", and "You are a freshman" respecively.

$$a \rightarrow (c \vee \neg f)$$

Propositional equivalences:

Definition (9)

A compound, proposition that is always true no matter what the truth value of the proposition variables that occur in it is called tautology.

Important note:
A compound proposition that is always false is called contradiction.
A compound that is neither a tautology nor a contradiction is called a contingency.

Definition (10)

The compound propositions p and q are called logically equivalent $p \equiv q$.

Important note:
The notation $p \equiv q$ denotes that p and q are logically equivalent.

Important logical equivalent rules:

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$

$p \to q \equiv \neg q \to \neg p$

$p \vee q \equiv \neg p \to q$

$p \wedge q \equiv \neg(p \to \neg q)$

$\neg(p \to q) \equiv p \wedge \neg q$

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Exercise (1)

p: it is below freezing.
q: It is snowing.

1. It is below freezing and snowing.

$$p \wedge q$$

2. It is below freezing but and snowing.

$$p \wedge \neg q$$

3. It is not below freezing and it is not snowing.

$$\neg p \wedge \neg q$$

4. It is either snowing or below freezing.

$$p \vee q$$

5. If it is below freezing, it is also snowing.

$$p \rightarrow q$$

6. Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.

$$(p \vee q) \wedge (\neg q \rightarrow p)$$

7. That it is below freezing is necessary and sufficient for it to be snowing.

$$p \leftrightarrow q$$

Exercise (2)

If we are not on vacation, then we go to the university and attend classes.

1. Write the proposition in symbolic form.

$$\neg p \rightarrow (q \wedge m)$$

2. Write the negation in symbolic form.

$$p \rightarrow \neg(q \wedge m)$$

Exercise (3)

Prove that $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ by using the truth table.

p	q	r	$(q \wedge r)$	$p \vee (q \wedge r)$	$(p \vee q)$	$(p \vee r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Shaded columns both have the same truth values which indicates that both statements are logically equivalent by using the truth table as a prove method.

Exercise (4)

Prove that $\mathbf{p \equiv \neg(p \wedge s) \rightarrow (\neg s \wedge p)}$ by using the equivalence rules.

R.H.S $\rightarrow \neg(p \wedge s) \rightarrow (\neg s \wedge p)$

$\equiv (\neg p \vee \neg s) \rightarrow (\neg s \wedge p)$

$\equiv \neg(\neg p \vee \neg s) \vee (\neg s \wedge p)$

$\equiv (p \wedge s) \vee (\neg s \wedge p)$

$\equiv p \wedge (s \vee \neg s)$

$\equiv p \wedge T$

$\equiv p$

Exercise (5)

Prove that $[(\mathbf{p} \rightarrow \mathbf{q}\ ) \wedge \neg \mathbf{q}] \rightarrow \neg \mathbf{p}$ is tautology.

$\equiv [(p \rightarrow q\ ) \wedge \neg q] \rightarrow \neg p$

$\equiv \neg\ [(p \rightarrow q\ ) \wedge \neg q] \vee \neg p$

$\equiv [\neg\ (p \rightarrow q\ ) \vee \neg\ (\neg q)] \vee \neg p$

$\equiv [\neg\ (\neg p \vee q\ ) \vee q] \vee \neg p$

$\equiv [p \wedge \neg q\ ) \vee q] \vee \neg p$

$\equiv [(p \vee q) \wedge (\neg q \vee q)] \vee \neg p$

$\equiv [(p \vee q) \wedge T] \vee \neg p$

$\equiv p \vee q \vee \neg p$

$\equiv (p \vee \neg p) \vee q$

$\equiv T \vee q$

$\equiv T$

Questions from Previous Exams

Question (1)

Let p: “Jake will sleep early” , q: “Jake will eat at home” , and r “it will rain”
Answer the following:

(a) Write in English :$\neg r \rightarrow \neg q$

If it will not rain, then Jack will not eat at home.

(b) Write in symbolic form:
A sufficient condition for either Jack eat at home or sleep early is it will rain.

$$r \rightarrow p \vee q$$

Question (2)

Let k : “The keys are lost”, d : “The door is closed”, and s :“The safe box is accessible”. Convert the following into symbolic form.

(a)Neither the keys are lost nor the door is closed.

$$\neg(k \vee d)$$
$$\equiv \neg k \wedge d$$

(b) If the keys are lost, then if the door is closed, then the safe box is not accessible.

$$k \rightarrow (d \rightarrow \neg s)$$

(c) The safe box is accessible if, and only if the door is not closed.

$$s \leftrightarrow \neg d$$

Question (3)

Show whether the following are logically equivalent or not.

$$p \to r \vee q \text{ and } p \wedge q \to r$$

p	q	r	r ∨ q	p → r ∨ q	p ∧ q	p ∧ q → r
T	T	T	T	T	T	T
T	T	F	T	T	T	F
T	F	T	T	T	F	T
T	F	F	F	F	F	T
F	T	T	T	T	F	T
F	T	F	T	T	F	T
F	F	T	T	T	F	T
F	F	F	F	T	F	T

The statements do not have the same truth values in the truth table which indicates that they are not logically equivalent.

Question (4)

Use Logical Equivalences to show that $(p \vee q) \wedge (r \vee (\neg p \wedge \neg q)) \to r$ is a tautology.

$(p \vee q) \wedge (r \vee (\neg p \wedge \neg q)) \to r \equiv (p \vee q) \wedge (r \vee \neg(p \vee q)) \to r$

$\equiv [(p \vee q) \wedge r] \vee [(p \vee q) \wedge \neg(p \vee q)] \to r$

$\equiv [(p \vee q) \wedge r] \vee F \to r$

$\equiv (p \vee q) \wedge r \to r$

$\equiv \neg(p \vee q) \vee \neg r \vee r$

$\equiv \neg(p \vee q) \vee T$

$\equiv T$