OL Academy

MATHS102
Calculus II

Lesson (1)
(4.4) L'Hopital's Rule

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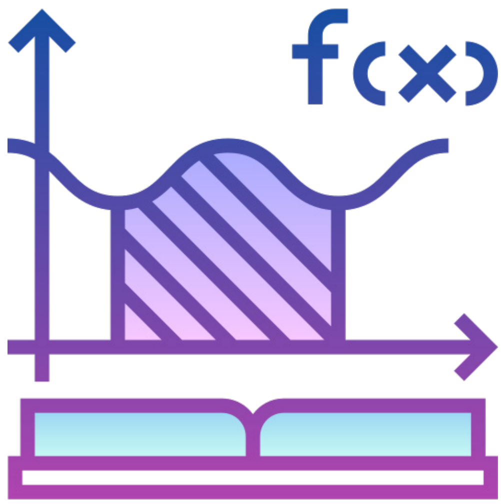

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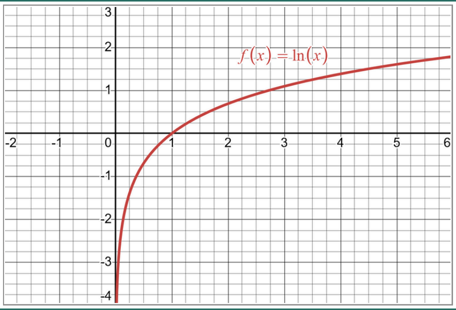
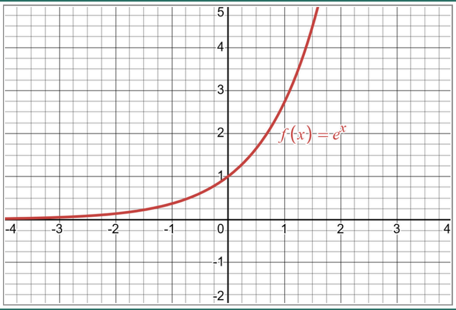
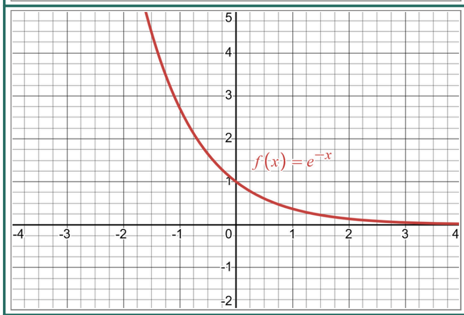
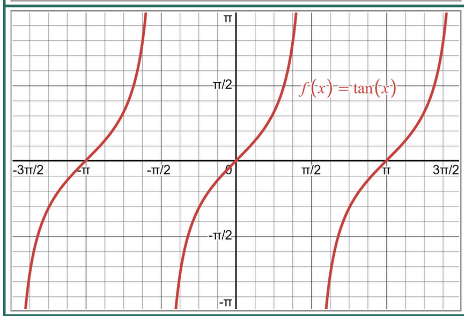
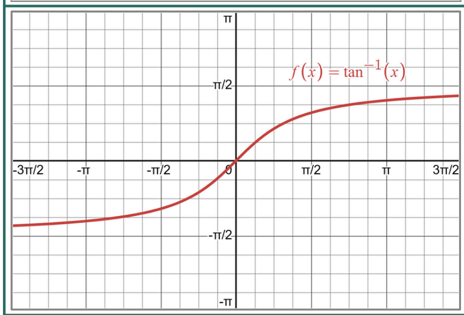
Function	Domain	Range	End Behavior
	$(0, \infty)$	$(-\infty, \infty)$	$\lim_{x\to\infty} \ln(x) = \infty$ $\lim_{x\to 0^+} \ln(x) = -\infty$
	$(-\infty, \infty)$	$(0, \infty)$	$\lim_{x\to\infty} e^x = \infty$ $\lim_{x\to-\infty} e^x = 0$
	$(-\infty, \infty)$	$(0, \infty)$	$\lim_{x\to\infty} e^{-x} = 0$ $\lim_{x\to-\infty} e^{-x} = \infty$
	$x \neq \frac{\pi}{2} + n\pi$	$(-\infty, \infty)$	$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$ $\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$
	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\lim_{x\to\pm\frac{\pi}{2}^+} \tan x = -\infty$ $\lim_{x\to\pm\frac{\pi}{2}^-} \tan x = \infty$
Exponential Functions $b^x, b > 1$	$(-\infty, \infty)$	$(0, \infty)$	$\lim_{x\to\infty} b^x = \infty$ $\lim_{x\to-\infty} b^x = 0$
Exponential Functions $b^x, 0 < b < 1$	$(-\infty, \infty)$	$(0, \infty)$	$\lim_{x\to\infty} b^x = 0$ $\lim_{x\to-\infty} b^x = \infty$

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Indeterminate Forms

Sometimes, direct substitution in a limit does not provide a clear answer. Instead, it leads to forms that require special techniques. These forms are called indeterminate forms. In this section, we will discuss these forms and how to evaluate the limits for each one correctly.

Indeterminate Fractions	Indeterminate Products	Indeterminate Differences	Indeterminate Powers
$\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$	$(0 \cdot \infty)$	$(\infty - \infty)$	$(0^0, \infty^0, 1^\infty)$

01 Indeterminate Fractions $\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$

$\lim_{x \to a} \frac{f(x)}{g(x)}$ → is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

(a) can be any real number or ∞

✔ **Apply L'Hospital's Rule:** $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

⚠ **Notice that, we can apply L'Hospital's Rule more than once.**

Examples

1 $\lim_{x \to 0} \frac{\sin x}{x} \longmapsto \frac{0}{0}$

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \cos(0) = 1$$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(x^n) = nx^{n-1}$

2 $\lim_{x \to \infty} \frac{e^x}{x^2} \longmapsto \frac{\infty}{\infty}$

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} \longmapsto \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{e^x}{2} = \infty$$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(x^n) = nx^{n-1}$

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(3) $\lim\limits_{x\to 0} \dfrac{x\cdot 3^x}{3^x - 1} \longmapsto \dfrac{0}{0}$

$$\lim_{x\to 0} \frac{x\cdot 3^x}{3^x - 1} = \lim_{x\to 0} \frac{3^x + x\cdot 3^x \ln(3)}{3^x \ln(3)}$$

$$= \lim_{x\to 0} \frac{3^x\left(1 + x\ln(3)\right)}{3^x \ln(3)}$$

$$= \lim_{x\to 0} \frac{1 + x\ln(3)}{\ln(3)} = \frac{1}{\ln(3)}$$

$\frac{d}{dx}(b^x) = b^x \ln(b), b > 0$

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Indeterminate Products $(0\cdot\infty)$

$\lim\limits_{x\to a} f(x)\cdot g(x)$ → is of the form $0\cdot\infty$

→ a can be any real number or ∞

- **Rewrite the Product as a Quotient:** $f(x)\cdot g(x) \to \dfrac{f(x)}{\frac{1}{g(x)}}$ *or* $\dfrac{g(x)}{\frac{1}{f(x)}}$
- **Apply L'Hospital's Rule:** $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$

Examples

(4) $\lim\limits_{x\to \frac{\pi}{2}^-} \tan x \ln(\sin x) \longmapsto 0\cdot\infty$

$$\lim_{x\to \frac{\pi}{2}^-} \tan x \ln(\sin x) = \lim_{x\to \frac{\pi}{2}^-} \frac{\ln(\sin x)}{\cot x}$$

$$= \lim_{x\to \frac{\pi}{2}^-} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x}$$

$$= \lim_{x\to \frac{\pi}{2}^-} \frac{\cos x}{\sin x}\cdot -\sin^2 x$$

$$= \lim_{x\to \frac{\pi}{2}^-} \left(-\cos x \sin x\right) = -\cos\left(\tfrac{\pi}{2}\right)\sin\left(\tfrac{\pi}{2}\right) = 0$$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$

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Indeterminate Differences $(\infty - \infty)$

$\lim_{x \to a} f(x) - g(x)$ → is of the form $\infty - \infty$

a → can be any real number or ∞

✔ **Rewrite the Difference as a Quotient or a Product:**

$$f(x) - g(x) \to \frac{A}{B} \text{ or } A \cdot B$$

✔ **You will get one of the previous forms**

Examples

⑤ $\lim_{x \to 0^+} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)$ ⟼ $\infty - \infty$

$$\lim_{x \to 0^+} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) = \lim_{x \to 0^+} \frac{x - e^x + 1}{x(e^x - 1)}$$

$$= \lim_{x \to 0^+} \frac{1 - e^x}{e^x + xe^x - 1}$$

$$= \lim_{x \to 0^+} \frac{-e^x}{e^x + e^x + xe^x}$$

$$= \lim_{x \to 0^+} \frac{-e^x}{e^x(2 + x)} = \lim_{x \to 0^+} \frac{-1}{2 + x} = -\frac{1}{2}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(f \cdot g) = f' \cdot g + f \cdot g'$$

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$\lim\limits_{x\to a} [f(x)]^{g(x)}$ → is of the form 0^0 or ∞^0 or 1^∞

→ can be any real number or ∞

✓ **Transform the power to logarithm:** $[f(x)]^{g(x)} \to y = \ln [f(x)]^{g(x)}$

$y = g(x) \ln f(x)$

✓ **Evaluate the limit of y and the main limit:**

$\lim\limits_{x\to a} y = L \Rightarrow \lim\limits_{x\to a} [f(x)]^{g(x)} = e^L$

Examples

6 $\lim\limits_{x\to\infty} (x)^{\frac{1}{x}} \longmapsto \infty^0$

$$y = \ln\left(x^{\frac{1}{x}}\right) = \frac{1}{x}\ln x$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1}{x}\ln x \longmapsto 0\cdot\infty$$

$$= \lim_{x\to\infty} \frac{\ln x}{x} \longmapsto \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{1}{\infty} = 0$$

$$\Rightarrow \lim_{x\to\infty} (x)^{\frac{1}{x}} = e^0 = 1$$

$$\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\ln\left(f^g\right) = g\ln\left(f\right)$$

⑦ $\lim_{x\to 1^-} (1-x)^{\ln x} \longmapsto 0^0$

$$y = \ln(1-x)^{\ln x} = \ln x \ln(1-x)$$

$$\lim_{x\to 1^-} y = \lim_{x\to 1^-} \ln x \ln(1-x) \longmapsto 0\cdot\infty$$

$$= \lim_{x\to 1^-} \frac{\ln(1-x)}{\frac{1}{\ln x}} \longmapsto \frac{\infty}{\infty}$$

$$= \lim_{x\to 1^-} \frac{-\frac{1}{1-x}}{-\frac{1}{x(\ln x)^2}}$$

$$= \lim_{x\to 1^-} \frac{x(\ln x)^2}{1-x} \longmapsto \frac{0}{0}$$

$$= \lim_{x\to 1^-} \frac{(\ln x)^2 + 2\ln x}{-1} = 0$$

$$\Rightarrow \lim_{x\to 1^-} (1-x)^{\ln x} = e^0 = 1$$

$$\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}$$
$$\frac{d}{dx}(x^n) = nx^{n-1}$$
$$\ln(f^g) = g\ln(f)$$
$$\frac{d}{dx}(f\cdot g) = f'\cdot g + f\cdot g'$$

⑧ $\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} \longmapsto 1^\infty$

$$y = \ln(1+x)^{\frac{1}{x}} = \frac{1}{x}\ln(1+x)$$

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} \frac{1}{x}\ln(1+x) \longmapsto 0\cdot\infty$$

$$= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \longmapsto \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = \frac{1}{1} = 1$$

$$\Rightarrow \lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e^1 = e$$

$$\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}$$
$$\frac{d}{dx}(x^n) = nx^{n-1}$$
$$\ln(f^g) = g\ln(f)$$

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Exercises

① $$\lim_{x\to 0^+} \frac{\tan^{-1}(2x)}{\ln x}$$

$$\lim_{x\to 0^+} \frac{\tan^{-1}(2x)}{\ln x} = \frac{\tan^{-1}(0)}{-\infty} = \frac{0}{-\infty} = 0$$

$$\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}$$
$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$
$$\ln\left(f^g\right) = g\ln\left(f\right)$$

② $$\lim_{x\to 0^+} \ln x - \ln(\sin x) \longmapsto \infty - \infty$$

$$\lim_{x\to 0^+} \ln x - \ln(\sin x) = \lim_{x\to 0^+} \ln\left(\frac{x}{\sin x}\right)$$

$$\lim_{x\to 0^+} \frac{x}{\sin x} \longmapsto \frac{0}{0}$$

$$\lim_{x\to 0^+} \frac{x}{\sin x} = \lim_{x\to 0^+} \frac{1}{\cos x} = \frac{1}{\cos(0)} = 1$$

$$\Rightarrow \lim_{x\to 0^+} \ln x - \ln(\sin x) = \ln(1) = 0$$

$$\frac{d}{dx}\left(\sin x\right) = \cos x$$
$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$
$$\ln(f) - \ln(g) = \ln\left(\frac{f}{g}\right)$$

③ $$\lim_{x\to\infty} \frac{\log_2 x}{\log_3(x+3)} \longmapsto \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{\log_2 x}{\log_3(x+3)} = \lim_{x\to\infty} \frac{\frac{1}{\ln(2)\,x}}{\frac{1}{\ln(3)\,(x+3)}}$$

$$= \lim_{x\to\infty} \frac{\ln(3)}{\ln(2)} \cdot \frac{x+3}{x} \longmapsto \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{\ln(3)}{\ln(2)} \cdot \frac{1}{1} = \frac{\ln(3)}{\ln(2)}$$

$$\frac{d}{dx}\left(\log_b[f(x)]\right) = \frac{f'(x)}{\ln(b)\,f(x)}$$
$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

④ $$\lim_{x\to 0} \frac{e^{ax}-1}{\sin x} \longmapsto \frac{0}{0}$$

$$\lim_{x\to 0} \frac{e^{ax}-1}{\sin x} = \lim_{x\to 0} \frac{ae^{ax}}{\cos x} = \frac{ae^0}{\cos(0)} = \frac{a}{1} = a$$

$$\frac{d}{dx}\left(\sin x\right) = \cos x$$
$$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$$

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⑤ **If** $\lim_{x\to 0} \frac{\sin(ax)+bx^3-2x}{x^3} = 0$.

Find a and b.

$$\lim_{x\to 0} \frac{\sin(ax)+bx^3-2x}{x^3} \longmapsto \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{a\cos(ax)+3bx^2-2}{3x^2} \longmapsto \frac{0}{0}$$

$$\Rightarrow a\cos(0)+3b(0)^2-2=0$$

$$\Rightarrow a-2=0$$

$$\Rightarrow a=2$$

$$= \lim_{x\to 0} \frac{2\cos(2x)+3bx^2-2}{3x^2} \longmapsto \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-4\sin(2x)+6bx}{6x} \longmapsto \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-8\cos(2x)+6b}{6} = 0$$

$$\Rightarrow \frac{-8\cos(0)+6b}{6} = 0$$

$$\Rightarrow -8\cos(0)+6b=0$$

$$\Rightarrow -8+6b=0$$

$$\Rightarrow 6b=8$$

$$\Rightarrow b=\frac{8}{6}=\frac{4}{3}$$

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