7.4 – Integration of Rational Functions

Partial fraction decomposition is a powerful method used to make the integration of rational functions "fraction where both the numerator and denominator are polynomials" easier. The idea behind this technique is to break down a complex rational function into simpler fractions which can be easily integrated.

In this section, we will show how to integrate any rational function by expressing it as a sum of simpler fractions "partial fractions" that we already know how to integrate.

Notice that,

$a^2 - b^2 = (a-b)(a+b)$

$a^2 + 2ab + b^2 = (a+b)^2$

$a^2 - 2ab + b^2 = (a-b)^2$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Suppose that we want to evaluate the integral $\int \frac{f(x)}{g(x)} dx$, where $f(x)$ and $g(x)$ are polynomial functions.

Key steps in the process:

(1) Check the degrees of the polynomials:

If $deg\big(f(x)\big) \geq deg\big(g(x)\big)$, then use a long division to obtain $\frac{f(x)}{g(x)} = E(x) + \frac{r(x)}{g(x)}$, where $E(x)$ is a new polynomial with $deg\big(f(x)\big) < deg\big(g(x)\big)$.

(2) Factor the denominator:

If $deg\big(f(x)\big) < deg\big(g(x)\big)$, we express $g(x)$ as a product of linear factors of the form $ax + b$ or quadratic factors of the form $ax^2 + bx + c$.

(3) Set Up the Partial Fraction Decomposition:

Decompose $\frac{f(x)}{g(x)}$ to partial fractions.

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(ax+b)^n\left(cx^2+dx+e\right)^m} = \frac{r_1}{ax+b} + \frac{r_2}{(ax+b)^2} + \cdots + \frac{r_n}{(ax+b)^n} + \frac{s_1x+t_1}{cx^2+dx+e} + \frac{s_2x+t_2}{\left(cx^2+dx+e\right)^2} + \cdots + \frac{s_mx+t_m}{\left(cx^2+dx+e\right)^m}$$

(4) Solve for the Constants.

To do this, multiply both sides of the equation by the denominator and then equate the resulting polynomial to the original numerator. This process gives a system of equations, which can be solved to determine the values of the constants.

(5) Integrate each term.

➢ Examples:

1. $I = \int \frac{x^3-6x^2+5x-3}{x^2-1}\,dx$

Step (1):

degree of the numerator $= 3 \;>\;$ degree of the denominator $= 2$

$$\begin{array}{r|l} & \qquad\quad x-6 \\ \hline x^2-1 & x^3-6x^2+5x-3 \\ & x^3 \qquad\quad\; -x \qquad\quad \ominus \\ \cline{2-2} & \qquad -6x^2+6x-3 \\ & \qquad -6x^2 \qquad +6 \quad \ominus \\ \cline{2-2} & \qquad\qquad\quad 6x-9 \end{array}$$

$$\Rightarrow \quad \frac{x^3-6x^2+5x-3}{x^2-1} = (x-6) + \frac{6x-9}{x^2-1}$$

Step (2):

$$\frac{6x-9}{x^2-1} = \frac{6x-9}{(x-1)(x+1)}$$

$$\Rightarrow \quad \frac{x^3-6x^2+5x-3}{x^2-1} = (x-6) + \frac{6x-9}{(x-1)(x+1)}$$

Step (3):

$$\frac{6x-9}{x^2-1} = \frac{6x-9}{(x-1)(x+1)} = \frac{a}{x-1} + \frac{b}{x+1}$$

$$\Rightarrow \quad \frac{x^3-6x^2+5x-3}{x^2-1} = (x-6) + \frac{a}{x-1} + \frac{b}{x+1}$$

Step (4):

$$\frac{6x-9}{x^2-1} = \frac{a}{x-1} + \frac{b}{x+1} = \frac{a(x+1)+b(x-1)}{x^2-1}$$

$$\Rightarrow \quad 6x-9 = a(x+1) + b(x-1)$$

$$6x-9 = ax + a + bx - b$$

First method:

$$x = 1 \quad \Rightarrow \quad 6-9 = a(1+1)+0 \quad \Rightarrow \quad -3 = 2a \quad \Rightarrow \quad a = -\frac{3}{2}$$

$$x = -1 \quad \Rightarrow \quad -6-9 = 0 + b(-1-1) \quad \Rightarrow \quad -15 = -2b \quad \Rightarrow \quad b = \frac{15}{2}$$

Second method:

"Compare the coefficients"

Coefficient of x: $\quad a + b = 6 \quad \longrightarrow \quad (1)$

Coefficient of x^0: $\quad a - b = -9 \quad \longrightarrow \quad (2)$

$(1)+(2) \Rightarrow \quad 2a = -3 \quad \Rightarrow \quad a = -\frac{3}{2}$

$(1)-(2) \Rightarrow \quad 2b = 15 \quad \Rightarrow \quad b = \frac{15}{2}$

$$\Longrightarrow \quad \frac{x^3-6x^2+5x-3}{x^2-1} = (x-6) + \frac{-3/2}{x-1} + \frac{15/2}{x+1}$$

Step (4):

$$I = \int (x-6) + \frac{-3/2}{x-1} + \frac{15/2}{x+1}\,dx$$

$$= \int x\,dx - 6\int dx - \frac{3}{2}\int \frac{1}{x-1}dx + \frac{15}{2}\int \frac{1}{x+1}dx$$

$$= \frac{1}{2}x^2 - 6x - \frac{3}{2}ln|x-1| + \frac{15}{2}ln|x+1| + C$$

2. $J = \int \frac{-2x+4}{(x^2+1)(x-1)^2}dx$

Step (1):

degree of the numerator = 1 $<$ degree of the denominator = 4

Step (2):

the denominator is already factored "expressed as a product of linear and quadratic factors"

Step (3):

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{cx+d}{x^2+1}$$

Step (4):

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{cx+d}{x^2+1} = \frac{a(x-1)(x^2+1)+b(x^2+1)+(cx+d)(x-1)^2}{(x^2+1)(x-1)^2}$$

$$\Rightarrow \quad -2x+4 = a(x-1)(x^2+1) + b(x^2+1) + (cx+d)(x-1)^2$$

$$-2x+4 = a(x^3+x-x^2-1) + b(x^2+1) + (cx+d)(x^2-2x+1)$$

$$-2x+4 = ax^3 + ax - ax^2 - a + bx^2 + b + cx^3 - 2cx^2 + cx + dx^2 - 2dx + d$$

$$-2x+4 = (a+c)x^3 + (-a+b-2c+d)x^2 + (a+c-2d)x + (-a+b+d)$$

"Compare the coefficients"

Coefficient of x^3: $a+c=0 \longrightarrow (1)$

Coefficient of x^2: $-a+b-2c+d=0 \longrightarrow (2)$

Coefficient of x: $a+c-2d=-2 \longrightarrow (3)$

Coefficient of x^0: $-a+b+d=4 \longrightarrow (4)$

$(1)-(3) \Rightarrow \quad 2d=2 \quad \Rightarrow \quad d=1$

$(2)-(4) \Rightarrow \quad -2c=-4 \quad \Rightarrow \quad c=2$

From $(1) \Rightarrow \quad a+2=0 \quad \Rightarrow \quad a=-2$

From $(4) \Rightarrow \quad b+3=4 \quad \Rightarrow \quad b=1$

$$\Longrightarrow \quad \frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{-2}{x-1} + \frac{1}{(x-1)^2} + \frac{2x+1}{x^2+1}$$

Step (4):

$$J=\int \frac{-2}{x-1}+\frac{1}{(x-1)^2}+\frac{2x+1}{x^2+1}\ dx$$

$$=\int \frac{-2}{x-1}+\frac{1}{(x-1)^2}+\frac{2x}{x^2+1}+\frac{1}{x^2+1}\ dx$$

$$=-2\int \frac{1}{x-1}dx+\int \frac{1}{(x-1)^2}dx+2\int \frac{x}{x^2+1}dx+\int \frac{1}{x^2+1}dx$$

$$=-2\,ln|x-1|-\frac{1}{x-1}+ln\left|x^2+1\right|+tan^{-1}x+C$$

Notice that:

$$\int \frac{1}{x^2+a^2}dx=\frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)+C$$

✿ **Proof:**

$$\int \frac{1}{x^2+a^2}dx=\int \frac{1}{a^2\left(\frac{x^2}{a^2}+1\right)}dx$$

$$=\frac{1}{a^2}\int \frac{1}{\left(\frac{x}{a}\right)^2+1}dx$$

Let $u=\frac{x}{a} \quad \Rightarrow \quad du=\frac{1}{a}\,dx \quad \Rightarrow \quad dx=a\,du$

$$=\frac{1}{a^2}\int \frac{1}{u^2+1}\cdot a\,du$$

$$=\frac{1}{a}\int \frac{1}{u^2+1}du=\frac{1}{a}tan^{-1}u+C=\frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)+C$$

7.4 – Integration of Rational Functions – Exercises

❖ **Write out the form of the partial fraction decomposition of the following functions. Do not determine the numerical values of the coefficients.**

1. $\frac{4+x}{(1+2x)(3-x)}$

   $\frac{4+x}{(1+2x)(3-x)} = \frac{a}{1+2x} + \frac{b}{3-x}$

2. $\frac{1}{x^2+x^4}$

   $\frac{1}{x^2+x^4} = \frac{1}{x^2(1+x^2)} = \frac{a}{x} + \frac{b}{x^2} + \frac{cx+d}{1+x^2}$

3. $\frac{1}{(x-1)^2(x^2+2)^2}$

   $\frac{1}{(x-1)^2(x^2+2)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{cx+d}{x^2+2} + \frac{ex+f}{(x^2+2)^2}$

4. $\frac{2x^2+5x+9}{(3x+2)^3(x+5)}$

   $\frac{2x^2+5x+9}{(3x+2)^3(x+5)} = \frac{a}{3x+2} + \frac{b}{(3x+2)^2} + \frac{c}{(3x+2)^3} + \frac{d}{x+5}$

5. $\frac{x-4}{x^2-5x+6}$

   $\frac{x-4}{x^2-5x+6} = \frac{x-4}{(x-2)(x-3)} = \frac{a}{x-2} + \frac{b}{x-3}$

6. $\frac{4x}{(x-1)^2(x+1)}$

   $\frac{4x}{(x-1)^2(x+1)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1}$

7. $\frac{3x+1}{(x^2+1)(x-1)(x+5)}$

   $\frac{3x+1}{(x^2+1)(x-1)(x+5)} = \frac{ax+b}{x^2+1} + \frac{c}{x-1} + \frac{d}{x+5}$

8. $\frac{2x^2-x+4}{x^3+4x}$

   $\frac{2x^2-x+4}{x^3+4x} = \frac{2x^2-x+4}{x(x^2+4)} = \frac{a}{x} + \frac{bx+c}{x^2+4}$

9. $\frac{1-x+2x^2-x^3}{x(x^2+1)^2}$

   $\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{a}{x} + \frac{bx+c}{x^2+1} + \frac{dx+e}{(x^2+1)^2}$

❖ **Evaluate the indefinite integral.**

10. $I = \int \frac{x-9}{(x+5)(x-2)}\,dx$

Step (1):

degree of the numerator $= 1 \quad < \quad$ degree of the denominator $= 2$

Step (2):

the denominator is already factored "expressed as a product of linear and quadratic factors"

Step (3):

$$\frac{x-9}{(x+5)(x-2)} = \frac{a}{x+5} + \frac{b}{x-2}$$

Step (4):

$$\frac{x-9}{(x+5)(x-2)} = \frac{a}{x+5} + \frac{b}{x-2} = \frac{a(x-2)+b(x+5)}{(x+5)(x-2)}$$

$$\Rightarrow \quad x-9 = a(x-2) + b(x+5)$$

$$x-9 = ax - 2a + bx + 5b$$

First method:

$x = 2 \quad \Rightarrow \quad 2-9 = 0 + b(2+5) \quad \Rightarrow \quad -7 = 7b \quad \Rightarrow \quad b = -1$

$x = -5 \quad \Rightarrow \quad -5-9 = a(-5-2) + 0 \quad \Rightarrow \quad -14 = -7a \quad \Rightarrow \quad a = 2$

Second method:

"Compare the coefficients"

Coefficient of x: $\quad a + b = 1 \quad \longrightarrow \quad (1)$

Coefficient of x^0: $\quad -2a + 5b = -9 \quad \longrightarrow \quad (2)$

$2(1) + (2) \Rightarrow \quad 7b = -7 \quad \Rightarrow \quad b = -1$

From $(1) \Rightarrow \quad a - 1 = 1 \quad \Rightarrow \quad a = 1 + 1 \quad \Rightarrow \quad a = 2$

$\int \frac{1}{x}\,dx = ln|x| + C$

$$\Longrightarrow \quad \frac{x-9}{(x+5)(x-2)} = \frac{2}{x+5} + \frac{-1}{x-2}$$

Step (4):

$$I = \int \frac{2}{x+5} + \frac{-1}{x-2}\,dx$$

$$= 2\int \frac{1}{x+5}\,dx - \int \frac{1}{x-2}\,dx$$

$$= 2\,ln|x+5| - ln|x-2| + C$$

11. $K = \int \frac{x^4+9x^2+x+2}{x^2+9}\,dx$

degree of the numerator $= 4 \quad > \quad$ degree of the denominator $= 2$

$$\begin{array}{r|l} & \qquad x^2 \\ \hline x^2+9 & x^4+9x^2+x+2 \quad \ominus \\ & x^4+9x^2 \\ \hline & \qquad\qquad x+2 \end{array}$$

$$\frac{x^4+9x^2+x+2}{x^2+9} = x^2 + \frac{x+2}{x^2+9}$$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

$\int \frac{1}{x}\,dx = ln|x| + C$

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right) + C$

$$K = \int x^2 + \frac{x+2}{x^2+9}\,dx = \int x^2\,dx + \int \frac{x}{x^2+9}\,dx + 2\int \frac{1}{x^2+9}\,dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}ln\left|x^2+9\right| + \frac{2}{3}tan^{-1}\left(\frac{x}{3}\right) + C$$

12. $L = \int \frac{10}{(x-1)(x^2+9)} dx$

$$\frac{10}{(x-1)(x^2+9)} = \frac{a}{x-1} + \frac{bx+c}{x^2+9} = \frac{a(x^2+9)+(bx+c)(x-1)}{(x-1)(x^2+9)}$$

$$\Rightarrow \quad 10 = a(x^2+9) + (bx+c)(x-1)$$

$$10 = ax^2 + 9a + bx^2 - bx + cx - c$$

Finding the constant a, b, c:

At $x = 1 \quad \Rightarrow \quad 10 = a(1+9) + 0 \quad \Rightarrow \quad 10 = 10a \quad \Rightarrow \quad a = 1$

"Compare the coefficients"

Coefficient of x^2: $\quad a + b = 0 \quad \longrightarrow (1)$

Coefficient of x: $\quad -b + c = 0 \quad \longrightarrow (2)$

Coefficient of x^0: $\quad 9a - c = 10 \quad \longrightarrow (3)$

From $(1) \Rightarrow \quad b = -a \quad \Rightarrow \quad b = -1$

From $(2) \Rightarrow \quad c = b \quad \Rightarrow \quad c = -1$

$$\Longrightarrow \quad \frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} + \frac{-x-1}{x^2+9}$$

$\int \frac{1}{x} dx = ln|x| + C$

$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + C$

$$L = \int \frac{1}{x-1} + \frac{-x-1}{x^2+9} \, dx$$

$$= \int \frac{1}{x-1} dx - \int \frac{x}{x^2+9} dx - \int \frac{1}{x^2+9} dx$$

$$= ln|x-1| - \frac{1}{2} ln|x^2+9| - \frac{1}{3} tan^{-1}\left(\frac{x}{3}\right) + C$$

13. $M = \int \frac{1}{x^3-1} dx$

$$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{a}{x-1} + \frac{bx+c}{x^2+x+1} = \frac{a(x^2+x+1)+(bx+c)(x-1)}{(x-1)(x^2+x+1)}$$

$$\Rightarrow \quad 1 = a(x^2+x+1) + (bx+c)(x-1)$$

$$1 = ax^2 + ax + a + bx^2 - bx + cx - c$$

Finding the constant a, b, c:

At $x = 1 \quad \Rightarrow \quad 1 = a(1+1+1) + 0 \quad \Rightarrow \quad 1 = 3a \quad \Rightarrow \quad a = \frac{1}{3}$

"Compare the coefficients"

Coefficient of x^2: $\quad a + b = 0 \quad \longrightarrow (1)$

Coefficient of x: $\quad a + b + c = 0 \quad \longrightarrow (2)$

Coefficient of x^0: $\quad a - c = 1 \quad \longrightarrow (3)$

From $(1) \Rightarrow \quad b = -a \quad \Rightarrow \quad b = -\frac{1}{3}$

From $(3) \Rightarrow \quad c = a - 1 \quad \Rightarrow \quad c = \frac{1}{3} - 1 \quad \Rightarrow \quad c = -\frac{2}{3}$

$$\Longrightarrow \quad \frac{1}{x^3-1} = \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1}$$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$\int \frac{1}{x} dx = ln|x| + C$

$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + C$

$$M = \int \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1} \, dx = \frac{1}{3} \int \frac{1}{x-1} dx - \frac{1}{3} \int \frac{x+2}{x^2+x+1} dx$$

We need to complete the square.

$$x^2+x+1 = x^2+x+\frac{1}{4}-\frac{1}{4}+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

Let $u = x+\frac{1}{2} \quad \Rightarrow \quad du = dx$

$$M = \frac{1}{3}\int \frac{1}{x-1}dx - \frac{1}{3}\int \frac{u+\frac{3}{2}}{u^2+\frac{3}{4}}dx$$

$$= \frac{1}{3}\int \frac{1}{x-1}dx - \frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}}dx - \frac{1}{3}\cdot\frac{3}{2}\int \frac{1}{u^2+\frac{3}{4}}dx$$

$$= \frac{1}{3}ln|x-1| - \frac{1}{6}ln\left|u^2+\frac{3}{4}\right| - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}tan^{-1}\left(\frac{u}{\sqrt{3}/2}\right)+C$$

$$= \frac{1}{3}ln|x-1| - \frac{1}{6}ln|x^2+x+1| - \frac{\sqrt{3}}{3}tan^{-1}\left(\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}}\right)+C$$

$$= \frac{1}{3}ln|x-1| - \frac{1}{6}ln|x^2+x+1| - \frac{\sqrt{3}}{3}tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

14. $P = \int \frac{2x+1}{x^2-7x+12}dx$

$$\frac{2x+1}{x^2-7x+12} = \frac{2x+1}{(x-3)(x-4)} = \frac{a}{x-3}+\frac{b}{x-4} = \frac{a(x-4)+b(x-3)}{(x-3)(x-4)}$$

$\Rightarrow \quad 2x+1 = a(x-4)+b(x-3)$

$2x+1 = ax-4a+bx-3b$

Finding the constant a, b:

At $x=3 \quad \Rightarrow \quad 2(3)+1 = a(3-4)+0 \quad \Rightarrow \quad 7=-a \quad \Rightarrow \quad a=-7$

At $x=4 \quad \Rightarrow \quad 2(4)+1 = 0+b(4-3) \quad \Rightarrow \quad b=9$

$\Rightarrow \quad \frac{2x+1}{x^2-7x+12} = \frac{-7}{x-3}+\frac{9}{x-4}$

$\int \frac{1}{x}dx = ln|x|+C$

$$P = \int \frac{-7}{x-3}+\frac{9}{x-4}\ dx = -7\int\frac{1}{x-3}dx + 9\int\frac{1}{x-4}dx$$

$$= -7\,ln|x-3| + 9\,ln|x-4| + C$$

15. $R = \int \frac{x+3}{2x^3-8x}dx$

$$\frac{x+3}{2x^3-8x} = \frac{x+3}{2x(x^2-4)} = \frac{x+3}{2x(x-2)(x+2)} = \frac{a}{2x}+\frac{b}{x-2}+\frac{c}{x+2} = \frac{a(x-2)(x+2)+b(2x)(x+2)+c(2x)(x-2)}{2x(x-2)(x+2)}$$

$\Rightarrow \quad x+3 = a(x-2)(x+2)+b(2x)(x+2)+c(2x)(x-2)$

Finding the constant a, b, c:

At $x=0 \quad \Rightarrow \quad 3=-4a \quad \Rightarrow \quad a=-\frac{3}{4}$

At $x=2 \quad \Rightarrow \quad 5=16b \quad \Rightarrow \quad b=\frac{5}{16}$

At $x=-2 \quad \Rightarrow \quad -1=-16c \quad \Rightarrow \quad c=\frac{1}{16}$

$a^2-b^2 = (a-b)(a+b)$

$\int \frac{1}{x}dx = ln|x|+C$

$$R = \int \frac{-\frac{3}{4}}{2x}+\frac{\frac{5}{16}}{x-2}+\frac{\frac{1}{16}}{x+2}dx = -\frac{3}{8}\int\frac{1}{x}dx + \frac{5}{16}\int\frac{1}{x-2}dx + \frac{1}{16}\int\frac{1}{x+2}dx$$

$$= -\frac{3}{8}ln|x| + \frac{5}{16}ln|x-2| + \frac{1}{16}ln|x+2| + C$$

16. $W = \int \frac{x^2}{(x-1)(x^2+2x+1)} dx$

$$\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{x^2}{(x-1)(x+1)^2} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{c}{(x+1)^2} = \frac{a(x+1)^2+b(x-1)(x+1)+c(x-1)}{(x-1)(x+1)^2}$$

$$\Rightarrow \quad x^2 = a(x+1)^2 + b(x-1)(x+1) + c(x-1)$$

$$x^2 = ax^2 + 2ax + a + bx^2 - b + cx - c$$

Finding the constant a, b, c:

At $x = 1 \quad \Rightarrow \quad 1 = 4a \quad \Rightarrow \quad a = \frac{1}{4}$

At $x = -1 \quad \Rightarrow \quad 1 = -2c \quad \Rightarrow \quad c = -\frac{1}{2}$

"Compare the coefficients"

Coefficient of x^2: $\quad a + b = 1 \quad \longrightarrow (1)$

From $(1) \Rightarrow \quad b = 1 - a \quad \Rightarrow \quad b = \frac{3}{4}$

$$\Longrightarrow \quad \frac{x^2}{(x-1)(x^2+2x+1)} = \frac{x^2}{(x-1)(x+1)^2} = \frac{\frac{1}{4}}{x-1} + \frac{-\frac{1}{2}}{x+1} + \frac{\frac{3}{4}}{(x+1)^2}$$

$$W = \int \frac{\frac{1}{4}}{x-1} + \frac{\frac{3}{4}}{x+1} + \frac{-\frac{1}{2}}{(x+1)^2} \; dx = \frac{1}{4}\int \frac{1}{x-1} dx + \frac{3}{4}\int \frac{1}{x+1} dx - \frac{1}{2}\int \frac{1}{(x+1)^2} dx$$

$$= \frac{1}{4} ln|x-1| + \frac{3}{4} ln|x+1| + \frac{1}{2(x+1)} + C$$

$a^2 - b^2 = (a-b)(a+b)$

$\int \frac{1}{x} dx = ln|x| + C$

$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$

17. $S = \int \frac{8x^2+8x+2}{(4x^2+1)^2} dx$

$$\frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{ax+b}{4x^2+1} + \frac{cx+d}{(4x^2+1)^2} = \frac{(ax+b)(4x^2+1)+(cx+d)}{(4x^2+1)^2}$$

$$\Rightarrow \quad 8x^2 + 8x + 2 = (ax+b)(4x^2+1) + (cx+d)$$

$$8x^2 + 8x + 2 = 4ax^3 + ax + 4bx^2 + b + cx + d$$

Finding the constant a, b, c, d:

"Compare the coefficients"

Coefficient of x^3: $\quad 4a = 0 \quad \Rightarrow \quad a = 0$

Coefficient of x^2: $\quad 4b = 8 \quad \Rightarrow \quad b = 2$

Coefficient of x: $\quad a + c = 8 \quad \Rightarrow \quad c = 8$

Coefficient of x^0: $\quad b + d = 2 \quad \Rightarrow \quad d = 0$

$$\Longrightarrow \quad \frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{2}{4x^2+1} + \frac{8x}{(4x^2+1)^2}$$

$\int \frac{1}{x^2+1} dx = tan^{-1}(x) + C$

$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$

$$S = \int \frac{2}{4x^2+1} + \frac{8x}{(4x^2+1)^2} dx = 2\int \frac{1}{4x^2+1} dx + 8\int \frac{x}{(4x^2+1)^2} dx = 2I + 8J$$

Let $u = 2x \quad \Rightarrow \quad du = 2\,dx \;\; in\, I$ and $\omega = 4x^2 + 1 \quad \Rightarrow \quad d\omega = 8x\,dx \;\; in\, J$

$$= \int \frac{1}{u^2+1} du + \int \frac{1}{\omega^2} d\omega$$

$$= tan^{-1}(u) - \frac{1}{\omega} + C$$

$$= tan^{-1}(2x) - \frac{1}{4x^2+1} + C$$

18. $N = \int \frac{x^4+81}{x(x^2+9)^2}dx$

$$\frac{x^4+81}{x(x^2+9)^2} = \frac{a}{x} + \frac{bx+c}{x^2+9} + \frac{dx+e}{(x^2+9)^2} = \frac{a(x^2+9)^2+(bx+c)(x)(x^2+9)+(dx+e)(x)}{x(x^2+9)^2}$$

$$\Rightarrow \quad x^4 + 81 = a(x^2+9)^2 + (bx+c)x(x^2+9) + (dx+e)$$

$$x^4 + 81 = ax^4 + 18ax^2 + 81a + bx^4 + 9bx^2 + cx^3 + 9cx + dx^2 + ex$$

Finding the constant a, b, c, d, e:

At $x = 0 \quad \Rightarrow \quad 81 = 81a \quad \Rightarrow \quad a = 1$

"Compare the coefficients"

Coefficient of x^4: $\quad a + b = 1 \quad \Rightarrow \quad b = 0$

Coefficient of x^3: $\quad c = 0$

Coefficient of x^2: $\quad 18a + 9b + d = 0 \quad \Rightarrow \quad d = -18$

Coefficient of x: $\quad 9c + e = 0 \quad \Rightarrow \quad e = 0$

$$\Rightarrow \quad \frac{x^4+81}{x(x^2+9)^2} = \frac{1}{x} + \frac{-18x}{(x^2+9)^2}$$

$\int \frac{1}{x}dx = ln|x| + C$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

$$N = \int \frac{1}{x} + \frac{-18x}{(x^2+9)^2}dx = \int \frac{1}{x}dx - 18\int \frac{x}{(x^2+9)^2}dx$$

Let $u = x^2 + 9 \quad \Rightarrow \quad du = 2x\,dx$

$$= \int \frac{1}{x}dx - 9\int \frac{1}{u^2}du$$

$$= ln|x| + \frac{9}{x} + C = ln|x| + \frac{9}{x^2+9} + C$$

19. $O = \int \frac{1}{x^4+x}dx$

$$\frac{1}{x^4+x} = \frac{1}{x(x^3+1)} = \frac{1}{x(x+1)(x^2-x+1)} = \frac{a}{x} + \frac{b}{x+1} + \frac{cx+d}{x^2-x+1} = \frac{a(x+1)(x^2-x+1)+b(x)(x^2-x+1)+(cx+d)(x)(x+1)}{x(x+1)(x^2-x+1)}$$

$$\Rightarrow \quad 1 = a(x+1)(x^2-x+1) + b(x)(x^2-x+1) + (cx+d)(x)(x+1)$$

$$1 = ax^3 + a + bx^3 - bx^2 + bx + cx^3 + cx^2 + dx^2 + dx$$

Finding the constant a, b, c, d:

At $x = 0 \quad \Rightarrow \quad a = 1$

At $x = -1 \quad \Rightarrow \quad 1 = -3b \quad \Rightarrow \quad b = -\frac{1}{3}$

"Compare the coefficients"

Coefficient of x^3: $\quad a + b + c = 0 \quad \Rightarrow \quad c = -\frac{2}{3}$

Coefficient of x: $\quad b + d = 0 \quad \Rightarrow \quad d = \frac{1}{3}$

$$\Rightarrow \quad \frac{1}{x^4+x} = \frac{1}{x(x+1)(x^2-x+1)} = \frac{1}{x} + \frac{-\frac{1}{3}}{x+1} + \frac{-\frac{2}{3}x+\frac{1}{3}}{x^2-x+1}$$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$\int \frac{1}{x}dx = ln|x| + C$

$$O = \int \frac{1}{x} + \frac{-\frac{1}{3}}{x+1} + \frac{-\frac{2}{3}x+\frac{1}{3}}{x^2-x+1}\,dx = \int \frac{1}{x}dx - \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{3}\int \frac{2x-1}{x^2-x+1}dx$$

$$= ln|x| - \frac{1}{3}ln|x+1| - \frac{1}{3}ln|x^2 - x + 1| + C$$

20. $T = \int \frac{x^4}{x^2-1}\,dx$

degree of the numerator $= 4 \;>\;$ degree of the denominator $= 2$

$$\begin{array}{r|l} & \quad x^2+1 \\ \hline x^2-1 & x^4 \\ & x^4-x^2 \qquad \ominus \\ \cline{2-2} & \quad x^2 \\ & \quad x^2-1 \qquad \ominus \\ \cline{2-2} & \qquad 1 \end{array}$$

$\frac{x^4}{x^2-1} = (x^2+1) + \frac{1}{x^2-1}$

$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{a}{x-1} + \frac{b}{x+1} = \frac{a(x+1)+b(x-1)}{(x-1)(x+1)}$

$\Rightarrow \quad 1 = a(x+1) + b(x-1)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$\int \frac{1}{x}dx = ln|x| + C$

Finding the constant a, b:

At $x = 1 \quad \Rightarrow \quad 1 = 2a \quad \Rightarrow \quad a = \frac{1}{2}$

At $x = -1 \quad \Rightarrow \quad 1 = -2b \quad \Rightarrow \quad b = -\frac{1}{2}$

$\Longrightarrow \quad \frac{x^4}{x^2-1} = (x^2+1) + \frac{1}{x^2-1} = x^2 + 1 + \frac{a}{x-1} + \frac{b}{x+1} = x^2 + 1 + \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}$

$$T = \int x^2 + 1 + \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}\,dx = \int x^2\,dx + \int dx + \frac{1}{2}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{x-1}\,dx$$

$$= \frac{1}{3}x^3 + x + \frac{1}{2}ln|x+1| - \frac{1}{2}ln|x-1| + C$$

21. $E = \int \frac{16x^3}{4x^2-4x+1}\,dx$

degree of the numerator $= 3 \;>\;$ degree of the denominator $= 2$

$$\begin{array}{r|l} & \qquad 4x+4 \\ \hline 4x^2-4x+1 & 16x^3 \\ & 16x^3-16x^2+4x \qquad \ominus \\ \cline{2-2} & \qquad 16x^2-4x \\ & \qquad 16x^2-16x+4 \qquad \ominus \\ \cline{2-2} & \qquad\qquad 12x-4 \end{array}$$

$\frac{16x^3}{4x^2-4x+1} = (4x+4) + \frac{12x-4}{4x^2-4x+1}$

$\frac{12x-4}{4x^2-4x+1} = \frac{12x-4}{(2x-1)^2} = \frac{a}{2x-1} + \frac{b}{(2x-1)^2} = \frac{a(2x-1)+b}{(2x-1)^2}$

$\Rightarrow \quad 12x - 4 = a(2x-1) + b$

$\Rightarrow \quad 12x - 4 = 2ax - a + b$

$\int \frac{1}{x}dx = ln|x| + C$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

Finding the constant a, b:

"Compare the coefficients"

Coefficient of x: $\quad 2a = 12 \quad \Rightarrow \quad a = 6$

Coefficient of x^0: $\quad -a + b = -4 \quad \Rightarrow \quad b = 2$

$\Longrightarrow \quad \frac{16x^3}{4x^2-4x+1} = (4x+4) + \frac{12x-4}{4x^2-4x+1} = 4x + 4 + \frac{a}{2x-1} + \frac{b}{(2x-1)^2} = 4x + 4 + \frac{6}{2x-1} + \frac{2}{(2x-1)^2}$

$$E = \int 4x + 4 + \frac{6}{2x-1} + \frac{2}{(2x-1)^2}\,dx$$
$$= 4\int x\,dx + 4\int dx + 6\int \frac{1}{2x-1}\,dx + 2\int \frac{1}{(2x-1)^2}\,dx$$
$$= 2x^2 + 4x + 3\,ln|2x-1| - \tfrac{1}{2x-1} + C$$

22. $D = \int \frac{2x^4}{x^3-x^2+x-1}\,dx$

degree of the numerator $= 4 \;>\;$ degree of the denominator $= 3$

$$\begin{array}{r|l}
 & \quad\quad 2x+2 \\
\hline
x^3-x^2+x-1 & 2x^4 \\
 & 2x^4-2x^3+2x^2-2x \quad \ominus \\
\cline{2-2}
 & \quad 2x^3-2x^2+2x \\
 & \quad 2x^3-2x^2+2x-2 \quad \ominus \\
\cline{2-2}
 & \quad\quad\quad\quad 2
\end{array}$$

$$\frac{2x^4}{x^3-x^2+x-1} = (2x+2) + \frac{2}{x^3-x^2+x-1}$$

To factor $x^3 - x^2 + x - 1$:

$$\begin{aligned}
x^3 - x^2 + x - 1 &= (x^3-1) - x^2 + x \\
&= (x-1)(x^2+x+1) - x(x-1) \\
&= (x-1)[x^2+x+1-x] \\
&= (x-1)(x^2+1)
\end{aligned}$$

$$\frac{2}{x^3-x^2+x-1} = \frac{2}{(x^2+1)(x-1)} = \frac{ax+b}{x^2+1} + \frac{c}{x-1} = \frac{(ax+b)(x-1)+c(x^2+1)}{(x^2+1)(x-1)}$$

$\Rightarrow \quad 2 = (ax+b)(x-1) + c(x^2+1)$

$\Rightarrow \quad 2 = ax^2 - ax + bx - b + cx^2 + c$

Finding the constant a, b, c:

At $x = 1 \quad \Rightarrow \quad 2 = 2c \quad \Rightarrow \quad c = 1$

"Compare the coefficients"

Coefficient of x^2: $\quad a + c = 0 \quad \Rightarrow \quad a = -1$

Coefficient of x: $\quad -a + b = 0 \quad \Rightarrow \quad b = -1$

$$\Longrightarrow \quad \frac{2x^4}{x^3-x^2+x-1} = (2x+2) + \frac{2}{x^3-x^2+x-1} = 2x + 2 + \frac{-x-1}{x^2+1} + \frac{1}{x-1}$$

$$D = \int 2x + 2 + \frac{-x-1}{x^2+1} + \frac{1}{x-1}\,dx$$
$$= 2\int x\,dx + 2\int dx - \int \frac{x+1}{x^2+1}\,dx + \int \frac{1}{x-1}\,dx$$
$$= 2\int x\,dx + 2\int dx - \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx + \int \frac{1}{x-1}\,dx$$
$$= x^2 + 2x - \tfrac{1}{2}\,ln|x^2+1| - tan^{-1}(x) + ln|x-1| + C$$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

$\int \frac{1}{x}\,dx = ln|x| + C$

$\int \frac{1}{x^2+1}\,dx = tan^{-1}(x) + C$

23. $A = \int \frac{e^{4t}+2e^{2t}-e^{t}}{e^{2t}+1}dt$

Let $u = e^{t} \quad \Rightarrow \quad du = e^{t}\,dt$

$$A = \int \frac{e^{t}(e^{3t}+2e^{t}-1)}{e^{2t}+1}dt = \int \frac{u^3+2u-1}{u^2+1}du$$

degree of the numerator $= 3 \;>\;$ degree of the denominator $= 2$

$$\begin{array}{r|l} & \quad u \\ \hline u^2+1 & u^3+2u+1 \\ & u^3+\;\;u \qquad \ominus \\ \hline & \qquad u+1 \end{array}$$

$\frac{u^3+2u-1}{u^2+1} = u + \frac{u+1}{u^2+1}$

$$A = \int u + \frac{u+1}{u^2+1}\,du = \int u\,du + \int \frac{u}{u^2+1}du + \int \frac{1}{u^2+1}du$$
$$= \frac{1}{2}u^2 + \frac{1}{2}ln\left|u^2+1\right| + tan^{-1}u + C$$
$$= \frac{1}{2}e^{2t} + \frac{1}{2}ln\left|e^{2t}+1\right| + tan^{-1}e^{t} + C$$

$\int \frac{1}{x}dx = ln|x| + C$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

$\int \frac{1}{x^2+1}dx = tan^{-1}(x) + C$

24. $F = \int \frac{sin\,\theta}{cos^2\theta+cos\,\theta-2}d\theta$

Let $u = cos\,\theta \quad \Rightarrow \quad du = -\,sin\,\theta\;\;d\theta$

$$F = -\int \frac{1}{u^2+u-2}d\theta$$

$\frac{1}{u^2+u-2} = \frac{1}{(u-1)(u+2)} = \frac{a}{u-1} + \frac{b}{u+2}$

$\Rightarrow \quad 1 = a(u+2) + b(u-1)$

Finding the constant a, b:

At $u = 1 \quad \Rightarrow \quad 1 = 3a \quad \Rightarrow \quad a = \frac{1}{3}$

At $u = -2 \quad \Rightarrow \quad 1 = -3b \quad \Rightarrow \quad b = -\frac{1}{3}$

$\Longrightarrow \quad \frac{1}{u^2+u-2} = \frac{a}{u-1} + \frac{b}{u+2} = \frac{\frac{1}{3}}{u-1} + \frac{-\frac{1}{3}}{u+2}$

$$F = -\int \frac{\frac{1}{3}}{u-1} + \frac{-\frac{1}{3}}{u+2}\,du = -\frac{1}{3}\int \frac{1}{u-1}du + \frac{1}{3}\int \frac{1}{u+2}du$$
$$= -\frac{1}{3}ln|u-1| + \frac{1}{3}ln|u+2| + C$$
$$= -\frac{1}{3}ln|cos\,\theta-1| + \frac{1}{3}ln|cos\,\theta+2| + C$$

$\frac{d}{dx}(cos\,x) = -\,sin\,x$

$\int \frac{1}{x}dx = ln|x| + C$

25. $B = \int \frac{\sqrt{x+4}}{x}\,dx$

Let $u = \sqrt{x+4} \quad \Rightarrow \quad u^2 = x+4 \quad \Rightarrow \quad x = u^2 - 4 \quad \Rightarrow \quad dx = 2u\,du$

$$B = \int \frac{u}{u^2-4}\cdot 2u\,du = 2\int \frac{u^2}{u^2-4}\,du$$
$$= 2\int \frac{u^2-4+4}{u^2-4}\,du$$
$$= 2\int \frac{u^2-4}{u^2-4} + \frac{4}{u^2-4}\,du$$
$$= 2\int du + 8\int \frac{1}{u^2-4}\,du$$

$\frac{1}{u^2-4} = \frac{1}{(u-2)(u+2)} = \frac{a}{u-2} + \frac{b}{u+2}$

$\Rightarrow \quad 1 = a(u+2) + b(u-2)$

$\int \frac{1}{x}\,dx = ln|x| + C$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

Finding the constant a, b:

At $u = 2 \quad \Rightarrow \quad 1 = 4a \quad \Rightarrow \quad a = \frac{1}{4}$

At $u = -2 \quad \Rightarrow \quad 1 = -4b \quad \Rightarrow \quad b = -\frac{1}{4}$

$\Longrightarrow \quad \frac{1}{u^2-4} = \frac{a}{u-2} + \frac{b}{u+2} = \frac{\frac{1}{4}}{u-2} + \frac{-\frac{1}{4}}{u+2}$

$$B = 2\int du + 8\int \frac{1}{u^2-4}\,du = 2\int du + 8\left[\int \frac{\frac{1}{4}}{u-2} + \frac{-\frac{1}{4}}{u+2}\,du\right]$$
$$= 2\int du + 2\int \frac{1}{u-2}\,du - 2\int \frac{1}{u+2}\,du$$
$$= 2u + 2\,ln|u-2| - 2\,ln|u+2| + C$$
$$= 2\sqrt{x+4} + 2\,ln\left|\sqrt{x+4}-2\right| - 2\,ln\left|\sqrt{x+4}+2\right| + C$$

❖ **Evaluate the definite integral.**

26. $J = \int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx$

$$J = \int_3^4 \frac{x^3-2x^2}{x^3-2x^2} - \frac{4}{x^3-2x^2}\,dx = \int_3^4 1\,dx - \int_3^4 \frac{4}{x^3-2x^2}\,dx$$

$\frac{4}{x^3-2x^2} = \frac{4}{x^2(x-2)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-2} = \frac{ax(x-2)+b(x-2)+cx^2}{x^2(x-2)}$

$\Rightarrow \quad 4 = ax(x-2) + b(x-2) + cx^2$

$\Rightarrow \quad 4 = ax^2 - 2ax + bx - 2b + cx^2$

Finding the constant a, b, c:

At $x = 2 \quad \Rightarrow \quad 4 = 4c \quad \Rightarrow \quad c = 1$

At $x = 0 \quad \Rightarrow \quad 4 = -2b \quad \Rightarrow \quad b = -2$

"Compare the coefficients"

Coefficient of x: $\quad -2a + b = 0 \quad \Rightarrow \quad a = -1$

$\Longrightarrow \quad \frac{4}{x^3-2x^2} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-2} = \frac{-1}{x} + \frac{-2}{x^2} + \frac{1}{x-2}$

$$J=\int_3^4 1\,dx-\int_3^4\frac{4}{x^3-2x^2}dx=\int_3^4 1\,dx-\int_3^4\frac{-1}{x}+\frac{-2}{x^2}+\frac{1}{x-2}dx$$

$$=\int_3^4 1\,dx+\int_3^4\frac{1}{x}dx+2\int_3^4\frac{1}{x^2}dx-\int_3^4\frac{1}{x-2}dx$$

$$=\left[x+ln|x|-\frac{2}{x}-ln|x-2|\right]_3^4$$

$$=\left[4+ln|4|-\tfrac{2}{4}-ln|4-2|\right]-\left[3+ln|3|-\tfrac{2}{3}-ln|3-2|\right]$$

$$=\left[\tfrac{7}{2}+ln|4|-ln|2|\right]-\left[\tfrac{7}{3}+ln|3|-ln|1|\right]$$

$$=\left[\tfrac{7}{2}+ln|2|\right]-\left[\tfrac{7}{3}+ln|3|\right]=\tfrac{7}{6}+ln\left|\tfrac{2}{3}\right|$$

27. $Q=\int_{-1}^{0}\frac{x^3}{x^2-2x+1}dx$

degree of the numerator $= 3 \;>\;$ degree of the denominator $= 2$

$$\begin{array}{r|l}
 & \quad x+2 \\ \hline
x^2-2x+1 & x^3 \\
 & x^3-2x^2+x \quad \ominus \\ \hline
 & \quad 2x^2-x \\
 & \quad 2x^2-4x+2 \quad \ominus \\ \hline
 & \qquad 3x-2
\end{array}$$

$\frac{x^3}{x^2-2x+1}=(x+2)+\frac{3x-2}{x^2-2x+1}$

$\frac{3x-2}{x^2-2x+1}=\frac{3x-2}{(x-1)^2}=\frac{a}{x-1}+\frac{b}{(x-1)^2}=\frac{a(x-1)+b}{(x-1)^2}$

$\Rightarrow\quad 3x-2=a(x-1)+b$

$\Rightarrow\quad 3x-2=ax-a+b$

Finding the constant a,b:

At $x=1 \quad\Rightarrow\quad 3(1)-2=b \quad\Rightarrow\quad b=1$

"Compare the coefficients"

Coefficient of x^0: $\quad -a+b=-2 \quad\Rightarrow\quad a=3$

$\Longrightarrow\quad \frac{3x-2}{x^2-2x+1}=\frac{a}{x-1}+\frac{b}{(x-1)^2}=\frac{3}{x-1}+\frac{1}{(x-1)^2}$

$\int\frac{1}{x}dx=ln|x|+C$

$\int x^n\,dx=\frac{1}{n+1}x^{n+1}+C$

$$Q=\int_{-1}^{0}(x+2)+\frac{x^3}{x^2-2x+1}dx=\int_{-1}^{0}x+2+\frac{3}{x-1}+\frac{1}{(x-1)^2}dx$$

$$=\int_{-1}^{0}x\,dx+2\int_{-1}^{0}dx+3\int_{-1}^{0}\frac{1}{x-1}dx+\int_{-1}^{0}\frac{1}{(x-1)^2}dx$$

$$=\left[\tfrac{1}{2}x^2\right]_{-1}^{0}+2[x]_{-1}^{0}+3[ln|x-1|]_{-1}^{0}-\left[\tfrac{1}{x-1}\right]_{-1}^{0}$$

$$=\left[0-\tfrac{1}{2}\right]+2[0+1]+3[ln|-1|-ln|2|]-\left[\tfrac{1}{-1}-\tfrac{1}{-2}\right]$$

$$=-\tfrac{1}{2}+2-3\,ln(2)+\tfrac{1}{2}=2-3\,ln(2)$$