# 7.3 – Trigonometric Substitutions

**Trigonometric substitution is a technique used to simplify integrals when the integrand involves square roots (such as finding the area of a circle or ellipse) or rational functions of polynomials. The idea is to make a substitution using trigonometric identities to simplify the expression.**

**There are 3 common trigonometric substitutions. In this section, we will discuss these cases and how to evaluate the integral for each one.**

**(1) If $f$ contains $\sqrt{a^2 - x^2}$:**

Set $x = a\sin\theta \quad \Rightarrow \quad dx = a\cos\theta\, d\theta$

**This substitution works because:**

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta$$

➢ **Examples:**

**1.** $I = \int \frac{x^2}{\sqrt{9-x^2}}\,dx$

Set $x = 3\sin\theta \quad \Rightarrow \quad dx = 3\cos\theta\, d\theta$

$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9(1 - \sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$$

$$I = \int \frac{(3\sin\theta)^2}{3\cos\theta}\cdot 3\cos\theta\, d\theta = \int 9\sin^2\theta\, d\theta$$
$$= 9\int \sin^2\theta\, d\theta$$
$$= 9\int \frac{1-\cos(2\theta)}{2}\,d\theta$$
$$= \frac{9}{2}\int 1 - \cos(2\theta)\, d\theta$$
$$= \frac{9}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C$$
$$= \frac{9}{2}\left[\theta - \frac{1}{2}(2\sin\theta\cos\theta)\right] + C$$
$$= \frac{9}{2}\left[\theta - \sin\theta\cos\theta\right] + C$$

$\frac{d}{dx}(\sin x) = \cos x$

$\sin^2 x = \frac{1-\cos(2x)}{2}$

$\cos^2 x = 1 - \sin^2 x$

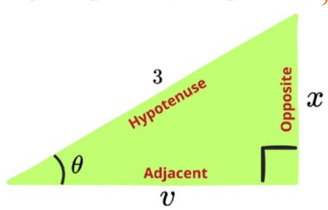
$\int cf(x)\,dx = c\int f(x)\,dx$

$\int \cos(cx)\,dx = \frac{1}{c}\sin(cx) + C$

$\sin(2x) = 2\sin x\cos x$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

- $x = 3\sin\theta \quad \Rightarrow \quad \sin\theta = \frac{x}{3} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{x}{3}\right)$
- To find $\sin\theta$ and $\cos\theta$, we use the following right triangle:




$$3^2 = x^2 + v^2 \quad \Rightarrow \quad v = \sqrt{9 - x^2}$$
$$\sin\theta = \frac{Opposite}{Hypotenuse} = \frac{x}{3}$$
$$\cos\theta = \frac{Adjacent}{Hypotenuse} = \frac{\sqrt{9-x^2}}{3}$$

$$I = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$$

2. $J = \int \frac{\sqrt{9-x^2}}{x^2} dx$

Set $x = 3\sin\theta \quad \Rightarrow \quad dx = 3\cos\theta\, d\theta$

$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$

$$J = \int \frac{3\cos\theta}{(3\sin\theta)^2} \cdot 3\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta} d\theta$$
$$= \int \cot^2\theta\, d\theta$$
$$= \int \csc^2\theta - 1\, d\theta$$
$$= -\cot\theta - \theta + C$$

$\frac{d}{dx}(\sin x) = \cos x$

$\cos^2 x = 1 - \sin^2 x$

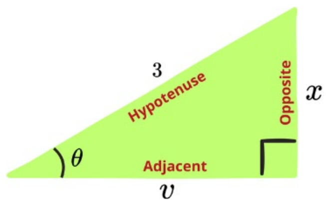
$\frac{\cos x}{\sin x} = \cot x$

$\cot^2 x = \csc^2 x - 1$

$\int \csc^2 x\, dx = -\cot x + C$

$\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C$

- $x = 3\sin\theta \quad \Rightarrow \quad \sin\theta = \frac{x}{3} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{x}{3}\right)$
- To find $\cot\theta$, we use the following right triangle:




$3^2 = x^2 + v^2 \quad \Rightarrow \quad v = \sqrt{9-x^2}$

$\cot\theta = \frac{Adjacent}{Opposite} = \frac{\sqrt{9-x^2}}{x}$

$$J = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

**(2) If $f$ contains $\sqrt{x^2-a^2}$:**

Set $x = a\sec\theta \quad \Rightarrow \quad dx = a\sec\theta\tan\theta\, d\theta$

**This substitution works because:**

$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta$

➢ **Examples:**

1. $S = \int \frac{\sqrt{x^2-1}}{x^4} dx$

Set $x = \sec\theta \quad \Rightarrow \quad dx = \sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$

$$S = \int \frac{\tan\theta}{\sec^4\theta} \cdot \sec\theta\tan\theta\, d\theta = \int \frac{\tan^2\theta}{\sec^3\theta} d\theta$$
$$= \int \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^3\theta\, d\theta$$
$$= \int \sin^2\theta\cos\theta\, d\theta$$

Let $u = \sin\theta \quad \Rightarrow \quad du = \cos\theta\, d\theta$

$$= \int u^2\, du = \frac{1}{3}u^3 + C$$
$$= \frac{1}{3}\sin^3\theta + C$$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\tan^2 x = \sec^2 x - 1$

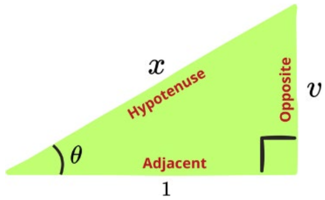
$\tan x = \frac{\sin x}{\cos x}$

$\cos x = \frac{1}{\sec x}$

$\frac{d}{dx}(\sin x) = \cos x$

$\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C$

- **To find $\sin\theta$, we use the following right triangle:**




$$x^2 = 1^2 + v^2 \quad \Rightarrow \quad v = \sqrt{x^2 - 1}$$

$$\sin\theta = \frac{Opposite}{Hypotenuse} = \frac{\sqrt{x^2-1}}{x}$$

$$S = \frac{1}{3}\sin^3\theta + C = \frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C = \frac{(x^2-1)^{\frac{3}{2}}}{3x^3} + C$$

**2.** $T = \int \frac{1}{\sqrt{25x^2-4}}dx$

$$T = \int \frac{1}{\sqrt{25\left(x^2 - \frac{4}{25}\right)}}dx = \frac{1}{5}\int \frac{1}{\sqrt{x^2 - \frac{4}{25}}}dx$$

Set $x = \frac{2}{5}\sec\theta \quad \Rightarrow \quad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta$

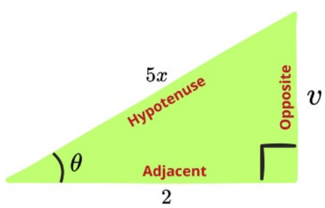
$$\sqrt{x^2 - \frac{4}{25}} = \sqrt{\frac{4}{25}\sec^2\theta - \frac{4}{25}} = \sqrt{\frac{4}{25}(\sec^2\theta - 1)} = \sqrt{\frac{4}{25}\tan^2\theta} = \frac{2}{5}\tan\theta$$

$$T = \frac{1}{5}\int \frac{1}{\frac{2}{5}\tan\theta}\cdot\frac{2}{5}\sec\theta\tan\theta\, d\theta = \frac{1}{5}\int \sec\theta\, d\theta$$

$$= \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

> $\frac{d}{dx}(\sec x) = \sec x \tan x$
>
> $\tan^2 x = \sec^2 x - 1$
>
> $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

- **To find $\sec\theta$ and $\tan\theta$, we use the following right triangle:**




$$(5x)^2 = 2^2 + v^2 \quad \Rightarrow \quad v = \sqrt{25x^2 - 4}$$

$$\sec\theta = \frac{Hypotenuse}{Adjacent} = \frac{5x}{2}$$

$$\tan\theta = \frac{Opposite}{Adjacent} = \frac{\sqrt{25x^2-4}}{2}$$

$$T = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C = \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C$$

**(3) If $f$ contains $\sqrt{x^2 + a^2}$:**

Set $x = a\tan\theta \quad \Rightarrow \quad dx = a\sec^2\theta\, d\theta$

**This substitution works because:**

$$\sqrt{x^2 + a^2} = \sqrt{a^2\tan^2\theta + a^2} = \sqrt{a^2(\tan^2\theta + 1)} = \sqrt{a^2\sec^2\theta} = a\sec\theta$$

➢ **Example:**

1. $L = \int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

Set $x = 2\tan\theta \quad \Rightarrow \quad dx = 2\sec^2\theta\, d\theta$

$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta} = 2\sec\theta$

$$L = \int \frac{1}{4\tan^2\theta \cdot 2\sec\theta} \cdot 2\sec^2\theta\, d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \sec\theta \cdot \cot^2\theta\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

Let $u = \sin\theta \quad \Rightarrow \quad du = \cos\theta\, d\theta$

$$= \frac{1}{4}\int \frac{1}{u^2}\, du = -\frac{1}{4u} + C$$

$$= -\frac{1}{4\sin\theta} + C$$

$$= -\frac{1}{4}\csc\theta + C$$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\sec^2 x = \tan^2 x + 1$

$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

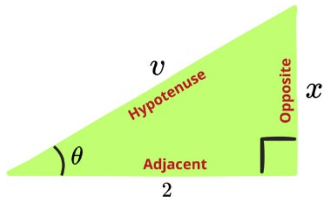
$\sec x = \frac{1}{\cos x}$

$\csc x = \frac{1}{\sin x}$

$\frac{d}{dx}(\sin x) = \cos x$

$\int x^n\, dx = \frac{1}{n+1}x^{n+1} + C$

✿ **To find $\csc\theta$, we use the following right triangle:**




$v^2 = x^2 + 4^2 \quad \Rightarrow \quad v = \sqrt{x^2+4}$

$\csc\theta = \frac{Hypotenuse}{Opposite} = \frac{\sqrt{x^2+4}}{x}$

$$S = -\frac{1}{4}\csc\theta + C = -\frac{1}{4}\left(\frac{\sqrt{x^2+4}}{x}\right) = -\frac{\sqrt{x^2+4}}{4x}$$

**(4) If $f$ contains $\sqrt{ax^2+bx+c}$:**

**To apply trigonometric substitution, we have to cancel the term $bx$ by completing the square.**

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2-4ac}{4a^2}\right]$$

➢ **Example:**

1. $Q = \int \frac{1}{\sqrt{x^2+2x+5}} dx$

$\sqrt{x^2+2x+5} = \sqrt{x^2+2x+1-1+5} = \sqrt{(x+1)^2+4}$

Let $u = x+1 \quad \Rightarrow \quad du = dx$

$$Q = \int \frac{1}{\sqrt{x^2+2x+5}} dx = \int \frac{1}{\sqrt{u^2+4}} du$$

Set $u = 2\tan\theta \quad \Rightarrow \quad du = 2\sec^2\theta \, d\theta$

$\sqrt{u^2+4} = \sqrt{4\tan^2\theta+4} = \sqrt{4(\tan^2\theta+1)} = \sqrt{4\sec^2\theta} = 2\sec\theta$

$$Q = \int \frac{1}{2\sec\theta} \cdot 2\sec^2\theta \, d\theta = \int \sec\theta \, d\theta$$

$$= ln|\sec\theta + \tan\theta| + C$$

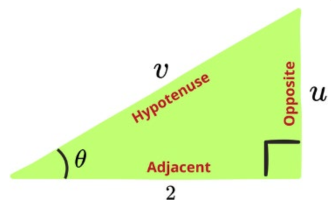
$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\sec^2 x = \tan^2 x + 1$

$\int \sec x \, dx = ln|\sec x + \tan x| + C$

❁ **To find $\sec\theta$ and $\tan\theta$, we use the following right triangle:**




$v^2 = u^2 + 4^2 \quad \Rightarrow \quad v = \sqrt{u^2+4}$

$\sec\theta = \frac{Hypotenuse}{Adjacent} = \frac{\sqrt{u^2+4}}{2}$

$\tan\theta = \frac{Opposite}{Adjacent} = \frac{u}{2}$

$$Q = ln|\sec\theta + \tan\theta| + C = ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right| + C$$

$$= ln\left|\frac{\sqrt{(x+1)^2+4}}{2} + \frac{x+1}{2}\right| + C$$

$$= ln\left|\frac{\sqrt{x^2+2x+5}}{2} + \frac{x+1}{2}\right| + C$$

# 7.3 – Trigonometric Substitutions – Exercises

❖ **Evaluate the indefinite integral.**

**1.** $L = \int \frac{(1-x^2)^{\frac{5}{2}}}{x^8} dx$

$$L = \int \frac{\left(\sqrt{1-x^2}\right)^5}{x^8} dx$$

Set $x = \sin\theta \quad \Rightarrow \quad dx = \cos\theta \, d\theta$

$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$$L = \int \frac{\cos^5\theta}{\sin^8\theta} \cdot \cos\theta \, d\theta = \int \frac{\cos^6\theta}{\sin^8\theta} d\theta$$

$$= \int \frac{\cos^6\theta}{\sin^6\theta} \cdot \frac{1}{\sin^2\theta} d\theta$$

$$= \int \cot^6\theta \csc^2\theta \, d\theta$$

Let $u = \cot\theta \quad \Rightarrow \quad du = -\csc^2\theta \, d\theta$

$$= -\int u^6 \, du = -\frac{1}{7}u^7 + C = -\frac{1}{7}\cot^7\theta + C$$

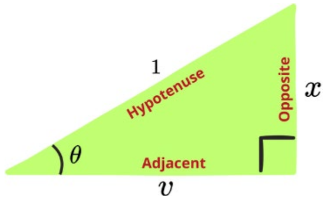
$\frac{d}{dx}(\sin x) = \cos x$

$\cos^2 x = 1 - \sin^2 x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C$

❁ To find $\cot\theta$, we use the following right triangle:




$1^2 = x^2 + v^2 \quad \Rightarrow \quad v = \sqrt{1-x^2}$

$\cot\theta = \frac{Adjacent}{Opposite} = \frac{\sqrt{1-x^2}}{x}$

$$\Longrightarrow L = -\frac{1}{7}\cot^7\theta + C = -\frac{1}{7}\left(\frac{\sqrt{1-x^2}}{x}\right)^7 + C = -\frac{1}{7}\frac{(1-x^2)^{\frac{7}{2}}}{x^7} + C$$

**2.** $M = \int \frac{1}{x^2\sqrt{x^2-16}} dx$

Set $x = 4\sec\theta \quad \Rightarrow \quad dx = 4\sec\theta\tan\theta \, d\theta$

$\sqrt{x^2-16} = \sqrt{16\sec^2\theta - 16} = \sqrt{16(\sec^2\theta - 1)} = \sqrt{16\tan^2\theta} = 4\tan\theta$

$$M = \int \frac{1}{16\sec^2\theta \cdot 4\tan\theta} \cdot 4\sec\theta\tan\theta \, d\theta$$

$$= \frac{1}{16}\int \frac{1}{\sec\theta} d\theta = \frac{1}{16}\int \cos\theta \, d\theta = \frac{1}{16}\sin\theta + C$$

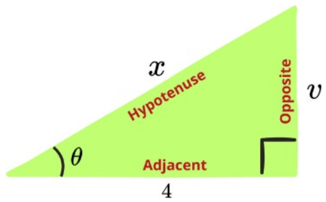
$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\tan^2 x = \sec^2 x - 1$

$\cos x = \frac{1}{\sec x}$

$\int \cos x \, dx = \sin x + C$

❁ To find $\sin\theta$, we use the following right triangle:




$x^2 = 4^2 + v^2 \quad \Rightarrow \quad v = \sqrt{x^2-16}$

$\sin\theta = \frac{Opposite}{Hypotenuse} = \frac{\sqrt{x^2-16}}{x}$

$$\Longrightarrow M = \frac{1}{16}\sin\theta + C = \frac{1}{16}\frac{\sqrt{x^2-16}}{x} + C$$

3. $I = \int \frac{x^2}{(3+4x-4x^2)^{\frac{3}{2}}} dx$

$3 + 4x - 4x^2 = -4\left(x^2 - x - \frac{3}{4}\right)$

$= -4\left(x^2 - x + \frac{1}{4} - \frac{1}{4} - \frac{3}{4}\right) = -4\left[\left(x - \frac{1}{2}\right)^2 - 1\right] = 4\left[1 - \left(x - \frac{1}{2}\right)^2\right]$

$I = \int \frac{x^2}{\left(\sqrt{3+4x-4x^2}\right)^3} dx = \int \frac{x^2}{\left(\sqrt{4[1-(x-1/2)^2]}\right)^3} dx = \frac{1}{8}\int \frac{x^2}{\left(\sqrt{1-(x-1/2)^2}\right)^3} dx$

Set $x - \frac{1}{2} = \sin\theta \quad \Rightarrow \quad dx = \cos\theta \, d\theta$

$\sqrt{1-(x-1/2)^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$I = \frac{1}{8}\int \frac{(\sin\theta + 1/2)^2}{\cos^3\theta} \cdot \cos\theta \, d\theta$

$= \frac{1}{8}\int \frac{\sin^2\theta + \sin\theta + 1/4}{\cos^2\theta} d\theta$

$= \frac{1}{8}\int \frac{\sin^2\theta}{\cos^2\theta} + \frac{\sin\theta}{\cos^2\theta} + \frac{1}{4\cos^2\theta} d\theta$

$= \frac{1}{8}\int \tan^2\theta + \frac{\sin\theta}{\cos^2\theta} + \frac{1}{4}\sec^2\theta \, d\theta$

$= \frac{1}{8}\int \sec^2\theta - 1 + \frac{\sin\theta}{\cos^2\theta} + \frac{1}{4}\sec^2\theta \, d\theta$

$= \frac{1}{8}\int \frac{5}{4}\sec^2\theta - 1 + \frac{\sin\theta}{\cos^2\theta} d\theta$

$= \frac{1}{8}\cdot\frac{5}{4}\int \sec^2\theta \, d\theta - \frac{1}{8}\int d\theta + \frac{1}{8}\int \frac{\sin\theta}{\cos^2\theta} d\theta$

Let $u = \cos\theta \quad \Rightarrow \quad du = -\sin\theta \, d\theta$

$= \frac{5}{32}\int \sec^2\theta \, d\theta - \int d\theta - \int \frac{1}{u^2} du = \frac{5}{32}\tan\theta - \frac{1}{8}\theta + \frac{1}{8u} + C$

$= \frac{5}{32}\tan\theta - \frac{1}{8}\theta + \frac{1}{8\cos\theta} + C$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = \sin x$

$\cos^2 x = 1 - \sin^2 x$

$\sec x = \frac{1}{\cos x}$

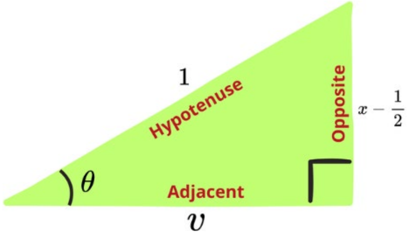
$\tan x = \frac{\sin x}{\cos x}$

$\tan^2 x = \sec^2 x - 1$

$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C$

$\int \sec^2 x \, dx = \tan x + C$

- $x - \frac{1}{2} = \sin\theta \quad \Rightarrow \quad \theta = \sin^{-1}\left(x - \frac{1}{2}\right)$
- To find $\cos\theta$ and $\tan\theta$, we use the following right triangle:




$1^2 = \left(x - \frac{1}{2}\right)^2 + v^2 \quad \Rightarrow \quad v = \sqrt{1-(x-1/2)^2}$

$\cos\theta = \frac{Adjacent}{Hypotenuse} = \sqrt{1-(x-1/2)^2}$

$\tan\theta = \frac{opposite}{Adjacent} = \frac{x-1/2}{\sqrt{1-(x-1/2)^2}}$

$\Longrightarrow I = \frac{5}{32}\tan\theta - \frac{1}{8}\theta + \frac{1}{8\cos\theta} + C$

$= \frac{5}{32}\frac{x-1/2}{\sqrt{1-(x-1/2)^2}} - \frac{1}{8}\sin^{-1}\left(x - \frac{1}{2}\right) + \frac{1}{8\sqrt{1-\left(x-\frac{1}{2}\right)^2}} + C$

4. $J=\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx$

Let $u=\ln x \quad \Rightarrow \quad du=\frac{1}{x}dx \quad \Rightarrow \quad dx = x\,du$

$$J=\int \frac{\sqrt{1-u^2}}{x\cdot u}\cdot x\,du=\int \frac{\sqrt{1-u^2}}{u}du$$

Set $u=\sin\theta \quad \Rightarrow \quad du=\cos\theta\,d\theta$

$$\sqrt{1-u^2}=\sqrt{1-\sin^2\theta}=\sqrt{\cos^2\theta}=\cos\theta$$

$$J=\int \frac{\cos\theta}{\sin\theta}\cdot\cos\theta\,d\theta=\int\frac{\cos^2\theta}{\sin\theta}d\theta$$
$$=\int\frac{1-\sin^2\theta}{\sin\theta}d\theta$$
$$=\int\frac{1}{\sin\theta}-\frac{\sin^2\theta}{\sin\theta}d\theta$$
$$=\int \csc\theta-\sin\theta\,d\theta$$
$$=\ln|\csc\theta-\cot\theta|+\cos\theta+C$$

$\frac{d}{dx}(\ln x)=\frac{1}{x}$

$\frac{d}{dx}(\sin x)=\cos x$

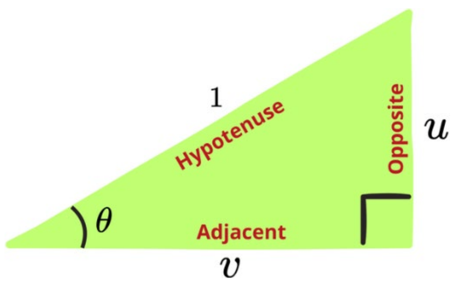
$\cos^2 x=1-\sin^2 x$

$\csc x=\frac{1}{\sin x}$

$\int \csc x\,dx=\ln|\csc x-\cot x|+C$

$\int \sin x\,dx=-\cos x+C$

❁ To find $\cot\theta$ , $\cos\theta$ and $\csc\theta$, we use the following right triangle:




$1^2=u^2+v^2 \quad \Rightarrow \quad v=\sqrt{1-u^2}$

$\cot\theta=\frac{Adjacent}{Opposite}=\frac{\sqrt{1-u^2}}{u}$

$\cos\theta=\frac{Adjacent}{Hypotenuse}=\sqrt{1-u^2}$

$\csc\theta=\frac{Hypotenuse}{Opposite}=\frac{1}{u}$

$$\Longrightarrow J=\ln|\csc\theta-\cot\theta|+\cos\theta+C=\ln\left|\frac{1}{u}-\frac{\sqrt{1-u^2}}{u}\right|+\sqrt{1-u^2}+C$$
$$=\ln\left|\frac{1}{\ln x}-\frac{\sqrt{1-(\ln x)^2}}{\ln x}\right|+\sqrt{1-(\ln x)^2}+C$$

5. $P=\int\sqrt{\frac{x}{1-x^3}}dx$

Let $u=x^{\frac{3}{2}} \quad \Rightarrow \quad du=\frac{3}{2}x^{\frac{1}{2}}dx \quad \Rightarrow \quad dx=\frac{2}{3\sqrt{x}}\,du$

$$P=\int\frac{\sqrt{x}}{\sqrt{1-x^3}}dx=\int\frac{\sqrt{x}}{\sqrt{1-(u^{2/3})^3}}\cdot\frac{2}{3\sqrt{x}}du$$
$$=\frac{2}{3}\int\frac{1}{\sqrt{1-u^2}}du$$
$$=\frac{2}{3}\sin^{-1}u+C$$
$$=\frac{2}{3}\sin^{-1}\left(x^{\frac{3}{2}}\right)+C$$
$$=\frac{2}{3}\sin^{-1}\left(\sqrt{x^3}\right)+C$$

$\frac{d}{dx}(x^n)=nx^{n-1}$

$\int\frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}x+C$

6. $Z = \int \frac{\sqrt{x^2-1}}{x^4} dx$

Set $x = \sec\theta \quad \Rightarrow \quad dx = \sec\theta \tan\theta \, d\theta$

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$

$$Z = \int \frac{\tan\theta}{\sec^4\theta} \cdot \sec\theta \tan\theta \, d\theta$$

$$= \int \frac{\tan^2\theta}{\sec^3\theta} d\theta = \int \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^3\theta \, d\theta$$

$$= \int \sin^2\theta \cos\theta \, d\theta$$

Let $u = \sin\theta \quad \Rightarrow \quad du = \cos\theta \, d\theta$

$$= \int u^2 \, du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3\theta + C$$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\tan^2 x = \sec^2 x - 1$

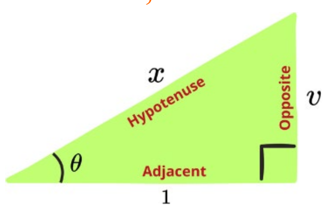
$\cos x = \frac{1}{\sec x}$

$\tan x = \frac{\sin x}{\cos x}$

$\frac{d}{dx}(\sin x) = \cos x$

$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C$

- To find $\sin\theta$, we use the following right triangle:




$x^2 = 1^2 + v^2 \quad \Rightarrow \quad v = \sqrt{x^2-1}$

$\sin\theta = \frac{Opposite}{Hypotenuse} = \frac{\sqrt{x^2-1}}{x}$

$$\Longrightarrow Z = \frac{1}{3}\sin^3\theta + C = \frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C = \frac{1}{3}\frac{\left(\sqrt{x^2-1}\right)^3}{x^3} + C = \frac{(x^2-1)^{\frac{3}{2}}}{3x^3} + C$$

7. $W = \int \frac{\sqrt{x^2-9}}{x^3} dx$

Set $x = 3\sec\theta \quad \Rightarrow \quad dx = 3\sec\theta\tan\theta \, d\theta$

$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9(\sec^2\theta - 1)} = \sqrt{9\tan^2\theta} = 3\tan\theta$

$$W = \int \frac{3\tan\theta}{27\sec^3\theta} \cdot 3\sec\theta\tan\theta \, d\theta$$

$$= \frac{9}{27}\int \frac{\tan^2\theta}{\sec^2\theta} d\theta = \frac{1}{3}\int \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta \, d\theta$$

$$= \frac{1}{6}\int 1 - \cos(2\theta) \, d\theta = \frac{1}{6}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C$$

$$= \frac{1}{6}[\theta - \sin\theta\cos\theta] + C$$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\tan^2 x = \sec^2 x - 1$

$\cos x = \frac{1}{\sec x}$

$\tan x = \frac{\sin x}{\cos x}$

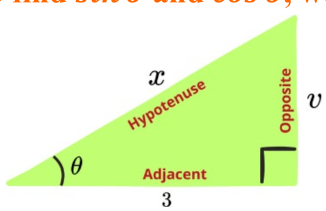
$\sin^2 x = \frac{1-\cos(2x)}{2}$

$\int \cos(cx) \, dx = \frac{1}{c}\sin(cx) + C$

$\sin(2x) = 2\sin x \cos x$

$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C$

- $x = 3\sec\theta \quad \Rightarrow \quad \sec\theta = \frac{x}{3} \quad \Rightarrow \quad \theta = \sec^{-1}\left(\frac{x}{3}\right)$
- To find $\sin\theta$ and $\cos\theta$, we use the following right triangle:




$x^2 = 3^2 + v^2 \quad \Rightarrow \quad v = \sqrt{x^2-9}$

$\sin\theta = \frac{Opposite}{Hypotenuse} = \frac{\sqrt{x^2-9}}{x}$

$\cos\theta = \frac{Adjacent}{Hypotenuse} = \frac{3}{x}$

$$\Longrightarrow W = \frac{1}{6}[\theta - \sin\theta\cos\theta] + C = \frac{1}{6}\left[\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{x} \cdot \frac{3}{x}\right] + C$$

**8.** $K = \int \frac{1}{\sqrt{4+x^2}} dx$

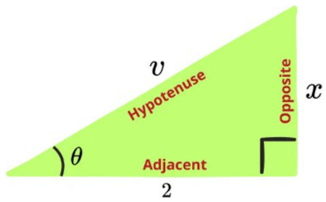
Set $x = 2\tan\theta \quad \Rightarrow \quad dx = 2\sec^2\theta\, d\theta$

$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta} = 2\sec\theta$

$$K = \int \frac{1}{2\sec\theta} \cdot 2\sec^2\theta\, d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

- To find $\sec\theta$ and $\tan\theta$, we use the following right triangle:




$v^2 = x^2 + 4^2 \quad \Rightarrow \quad v = \sqrt{x^2+4}$

$\sec\theta = \frac{Hypotenuse}{Adjacent} = \frac{\sqrt{x^2+4}}{2}$

$\tan\theta = \frac{Opposite}{Adjacent} = \frac{x}{2}$

$$\Longrightarrow K = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\sec^2 x = \tan^2 x + 1$

$\int \sec x\, dx = \ln|\sec\theta + \tan\theta| + C$

**9.** $S = \int x\sqrt{1-x^4}\, dx$

Let $u = x^2 \quad \Rightarrow \quad du = 2x\, dx \quad \Rightarrow \quad dx = \frac{1}{2x} du$

$$S = \int x\sqrt{1-u^2} \cdot \frac{1}{2x} du = \frac{1}{2}\int \sqrt{1-u^2}\, du$$

Set $u = \sin\theta \quad \Rightarrow \quad du = \cos\theta\, d\theta$

$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$$S = \frac{1}{2}\int \cos\theta \cdot \cos\theta\, d\theta = \frac{1}{2}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{4}\int 1 + \cos(2\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C$$

$$= \frac{1}{4}[\theta + \sin\theta\cos\theta] + C$$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(\sin x) = \cos x$

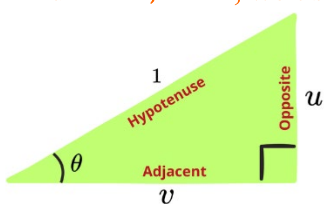
$\cos^2 x = 1 - \sin^2 x$

$\cos^2 x = \frac{1+\cos(2x)}{2}$

$\int \cos(cx)\, dx = \frac{1}{c}\sin(cx) + C$

$\sin(2x) = 2\sin x\cos x$

- $u = \sin\theta \quad \Rightarrow \quad \sin\theta = \frac{u}{1} \quad \Rightarrow \quad \theta = \sin^{-1}(u)$
- To find $\sin\theta$, $\cos\theta$, we use the following right triangle:




$1^2 = u^2 + v^2 \quad \Rightarrow \quad v = \sqrt{1-u^2}$

$\sin\theta = \frac{Opposite}{Hypotenuse} = \sqrt{1-u^2}$

$\cos\theta = \frac{Adjacent}{Hypotenuse} = u$

$$\Longrightarrow S = \frac{1}{4}[\theta + \sin\theta\cos\theta] + C = \frac{1}{4}\left[\sin^{-1}u + \sqrt{1-u^2} \cdot u\right] + C$$

$$= \frac{1}{4}\left[\sin^{-1}(x^2) + x^2\sqrt{1-x^4}\right] + C$$

**10.** $A = \int \frac{\cos x}{\sqrt{1+\sin^2 x}} dx$

Let $u = \sin x \quad \Rightarrow \quad du = \cos x\, dx$

$$A = \int \frac{\cos x}{\sqrt{1+u^2}} \cdot \cos x\, du = \int \frac{1}{\sqrt{1+u^2}} du$$

Set $u = \tan\theta \quad \Rightarrow \quad du = \sec^2\theta\, d\theta$

$\sqrt{u^2+1} = \sqrt{\tan^2\theta+1} = \sqrt{\sec^2\theta} = \sec\theta$

$$A = \int \frac{1}{\sec\theta} \cdot \sec^2\theta\, d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

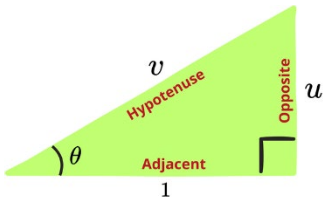
$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\sec^2 x = \tan^2 x + 1$

$\int \sec x\, dx = \ln|\sec\theta + \tan\theta| + C$

- To find $\sec\theta$ and $\tan\theta$, we use the following right triangle:




$v^2 = u^2 + 1^2 \quad \Rightarrow \quad v = \sqrt{u^2+1}$

$\sec\theta = \frac{Hypotenuse}{Adjacent} = \sqrt{u^2+1}$

$\tan\theta = \frac{Opposite}{Adjacent} = u$

$$\Longrightarrow A = \ln|\sec\theta + \tan\theta| + C = \ln\left|\sqrt{u^2+1} + u\right| + C = \ln\left|\sqrt{\sin^2\theta+1} + \sin\theta\right| + C$$

❖ **Evaluate the definite integral.**

**11.** $N = \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1+\sin^2 t}} dt$

Let $u = \sin t \quad \Rightarrow \quad du = \cos t\, dt$

When $t = 0 \quad \Rightarrow \quad u = \sin(0) = 0$

$t = \frac{\pi}{2} \quad \Rightarrow \quad u = \sin\left(\frac{\pi}{2}\right) = 1$

$$N = \int_0^1 \frac{\cos t}{\sqrt{1+u^2}} \cdot \cos t\, du = \int_0^1 \frac{1}{\sqrt{1+u^2}} du$$

Set $u = \tan\theta \quad \Rightarrow \quad du = \sec^2\theta\, d\theta$

When $u = 0 \quad \Rightarrow \quad \tan\theta = 0 \quad \Rightarrow \quad \theta = \tan^{-1}(0) = 0$

$u = 1 \quad \Rightarrow \quad \tan\theta = 1 \quad \Rightarrow \quad \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$\sqrt{u^2+1} = \sqrt{\tan^2\theta+1} = \sqrt{\sec^2\theta} = \sec\theta$

$$N = \int_0^{\frac{\pi}{4}} \frac{1}{\sec\theta} \cdot \sec^2\theta\, d\theta = \int_0^{\frac{\pi}{4}} \sec\theta\, d\theta$$

$$= [\ln|\sec\theta + \tan\theta|]_0^{\frac{\pi}{4}}$$

$$= \ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - \ln|\sec(0) + \tan(0)|$$

$$= \ln\left|\sqrt{2}+1\right| - \ln|1+0|$$

$$= \ln\left(\sqrt{2}+1\right) - \ln(1) = \ln\left(\sqrt{2}+1\right)$$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\sec^2 x = \tan^2 x + 1$

$\int \sec x\, dx = \ln|\sec\theta + \tan\theta| + C$

12. $O = \int_0^1 \sqrt{x - x^2}\, dx$

$$x - x^2 = -(x^2 - x) = -\left[x^2 - x + \frac{1}{4} - \frac{1}{4}\right] = -\left[\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}\right]$$
$$= \frac{1}{4} - \left(x - \frac{1}{2}\right)^2$$

Set $x - \frac{1}{2} = \frac{1}{2}\sin\theta \quad \Rightarrow \quad dx = \frac{1}{2}\cos\theta\, d\theta$

When $x = 0 \quad \Rightarrow \quad 0 - \frac{1}{2} = \frac{1}{2}\sin(\theta) \;\Rightarrow\; \sin(\theta) = -1 \;\Rightarrow\; \theta = \sin^{-1}(-1) = -\frac{\pi}{2}$

$x = 1 \quad \Rightarrow \quad 1 - \frac{1}{2} = \frac{1}{2}\sin(\theta) \;\Rightarrow\; \sin(\theta) = 1 \;\Rightarrow\; \theta = \sin^{-1}(1) = \frac{\pi}{2}$

$$\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} - \frac{1}{4}\sin^2\theta} = \sqrt{\frac{1}{4}(1 - \sin^2\theta)} = \sqrt{\frac{1}{4}\cos^2\theta} = \frac{1}{2}\cos\theta$$

$$O = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos\theta \cdot \frac{1}{2}\cos\theta\, d\theta = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$
$$= \frac{1}{4}\cdot\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + \cos(2\theta)\, d\theta$$
$$= \frac{1}{8}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= \frac{1}{8}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin(\pi)\right) - \left(-\frac{\pi}{2} + \frac{1}{2}\sin(-\pi)\right)\right]$$
$$= \frac{1}{8}\left[\left(\frac{\pi}{2} + 0\right) - \left(-\frac{\pi}{2} + 0\right)\right] = \frac{1}{8}\left(\frac{2\pi}{2}\right) = \frac{\pi}{8}$$

$\frac{d}{dx}(\sin x) = \cos x$

$\cos^2 x = 1 - \sin^2 x$

$\cos^2 x = \frac{1+\cos(2x)}{2}$

$\int x^n\, dx = \frac{1}{n+1}x^{n+1} + C$

$\int \cos(cx)\, dx = \frac{1}{c}\sin(cx) + C$

13. $Q = \int_{\ln\left(\frac{3}{4}\right)}^{\ln\left(\frac{4}{3}\right)} \frac{e^t}{(1+e^{2t})^{\frac{3}{2}}}\, dt$

Let $u = e^t \quad \Rightarrow \quad du = e^t dt$

When $t = \ln\left(\frac{3}{4}\right) \quad \Rightarrow \quad u = e^{\ln\left(\frac{3}{4}\right)} = \frac{3}{4}$

$t = \ln\left(\frac{4}{3}\right) \quad \Rightarrow \quad u = e^{\ln\left(\frac{4}{3}\right)} = \frac{4}{3}$

$$Q = \int_{\frac{3}{4}}^{\frac{4}{3}} \frac{e^t}{(1+u^2)^{\frac{3}{2}}}\cdot\frac{1}{e^t}\, du = \int_{\frac{3}{4}}^{\frac{4}{3}} \frac{1}{\left(\sqrt{1+u^2}\right)^3}\, du$$

Set $u = \tan\theta \quad \Rightarrow \quad du = \sec^2\theta\, d\theta$

When $u = \frac{3}{4} \quad \Rightarrow \quad \tan\theta = \frac{3}{4} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{4}\right)$

$u = \frac{4}{3} \quad \Rightarrow \quad \tan\theta = \frac{4}{3} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{4}{3}\right)$

$$\sqrt{u^2 + 1} = \sqrt{\tan^2\theta + 1} = \sqrt{\sec^2\theta} = \sec\theta$$

$$Q = \int_{\tan^{-1}\left(\frac{3}{4}\right)}^{\tan^{-1}\left(\frac{4}{3}\right)} \frac{1}{\sec^3\theta}\cdot\sec^2\theta\, d\theta = \int_{\tan^{-1}\left(\frac{3}{4}\right)}^{\tan^{-1}\left(\frac{4}{3}\right)} \frac{1}{\sec\theta}\, d\theta = \int_{\tan^{-1}\left(\frac{3}{4}\right)}^{\tan^{-1}\left(\frac{4}{3}\right)} \cos\theta\, d\theta$$
$$= [\sin\theta]_{\tan^{-1}\left(\frac{3}{4}\right)}^{\tan^{-1}\left(\frac{4}{3}\right)} = \frac{1}{5}$$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\sec^2 x = \tan^2 x + 1$

$\cos x = \frac{1}{\sec x}$

$\int \cos x\, dx = \sin x + C$

**14.** $T = \int_1^e \frac{1}{y\sqrt{1+(\ln y)^2}}\,dy$

Let $u = \ln y \quad \Rightarrow \quad du = \frac{1}{y}\,dy$

When $t = 1 \quad \Rightarrow \quad u = \ln(1) = 0$

$t = e \quad \Rightarrow \quad u = \ln(e) = 1$

$$T = \int_0^1 \frac{1}{y\sqrt{1+u^2}} \cdot y\,du = \int_0^1 \frac{1}{\sqrt{1+u^2}}\,du$$

Set $u = \tan\theta \quad \Rightarrow \quad du = \sec^2\theta\,d\theta$

When $u = 0 \quad \Rightarrow \quad \tan\theta = 0 \quad \Rightarrow \quad \theta = \tan^{-1}(0) = 0$

$u = 1 \quad \Rightarrow \quad \tan\theta = 1 \quad \Rightarrow \quad \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$\sqrt{u^2+1} = \sqrt{\tan^2\theta + 1} = \sqrt{\sec^2\theta} = \sec\theta$

$$T = \int_0^{\frac{\pi}{4}} \frac{1}{\sec\theta} \cdot \sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \sec\theta\,d\theta$$

$$= [\ln|\sec\theta + \tan\theta|]_0^{\frac{\pi}{4}}$$

$$= \ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - \ln|\sec(0) + \tan(0)|$$

$$= \ln|\sqrt{2}+1| - \ln|1+0| = \ln(\sqrt{2}+1)$$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\sec^2 x = \tan^2 x + 1$

$\int \sec x\,dx = \ln|\sec\theta + \tan\theta| + C$

**15.** $R = \int_0^{\frac{2}{3}} \sqrt{4-9x^2}\,dx$

$$R = \int_0^{\frac{2}{3}} \sqrt{4-9x^2}\,dx = \int_0^{\frac{2}{3}} \sqrt{9\left(\frac{4}{9}-x^2\right)}\,dx = 3\int_0^{\frac{2}{3}} \sqrt{\frac{4}{9}-x^2}\,dx$$

Set $x = \frac{2}{3}\sin\theta \quad \Rightarrow \quad dx = \frac{2}{3}\cos\theta\,d\theta$

When $x = 0 \quad \Rightarrow \quad 0 = \frac{2}{3}\sin(\theta) \quad \Rightarrow \quad \sin(\theta) = 0 \quad \Rightarrow \quad \theta = \sin^{-1}(0) = 0$

$x = \frac{2}{3} \quad \Rightarrow \quad \frac{2}{3} = \frac{2}{3}\sin(\theta) \quad \Rightarrow \quad \sin(\theta) = 1 \quad \Rightarrow \quad \theta = \sin^{-1}(1) = \frac{\pi}{2}$

$\sqrt{\frac{4}{9}-x^2} = \sqrt{\frac{4}{9}-\frac{4}{9}\sin^2\theta} = \sqrt{\frac{4}{9}(1-\sin^2\theta)} = \sqrt{\frac{4}{9}\cos^2\theta} = \frac{2}{3}\cos\theta$

$$R = 3\int_0^{\frac{\pi}{2}} \frac{2}{3}\cos\theta \cdot \frac{2}{3}\cos\theta\,d\theta = \frac{4}{3}\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= \frac{4}{6}\int_0^{\frac{\pi}{2}} 1 + \cos(2\theta)\,d\theta$$

$$= \frac{2}{3}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{3}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin(\pi)\right) - \left(0 + \frac{1}{2}\sin(0)\right)\right]$$

$$= \frac{2}{3}\left[\left(\frac{\pi}{2} + 0\right) - (0)\right] = \frac{\pi}{3}$$

$\frac{d}{dx}(\sin x) = \cos x$

$\cos^2 x = 1 - \sin^2 x$

$\cos^2 x = \frac{1+\cos(2x)}{2}$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

$\int \cos(cx)\,dx = \frac{1}{c}\sin(cx) + C$