OL Academy

MATHS101

Lesson 6

2.6 Limit at Infinity

2.6 Limits at Infinity; Horizontal Asymptotes

$\boxed{1}$ **Intuitive Definition of a Limit at Infinity** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

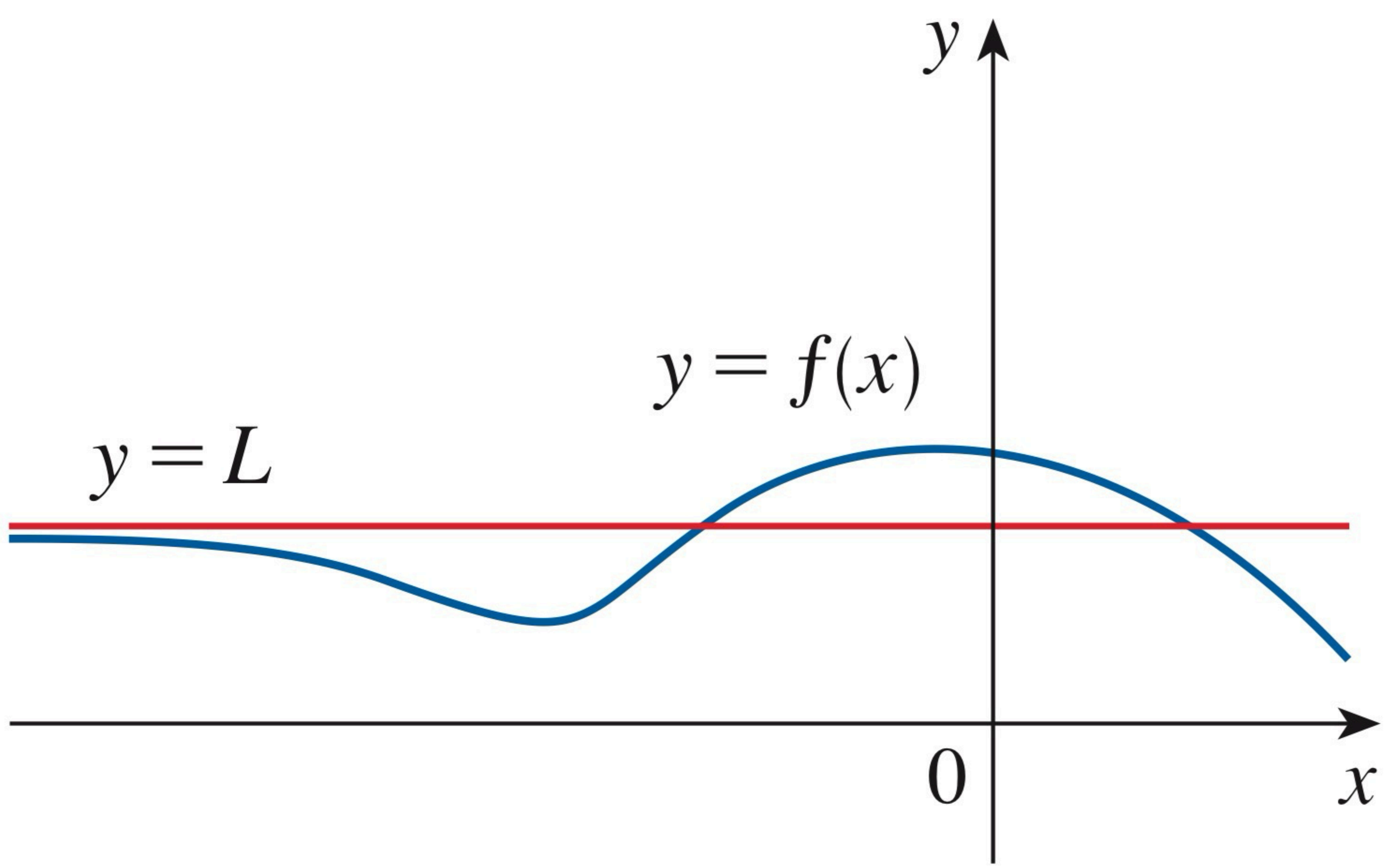
means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Another notation for $\lim_{x \to \infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$



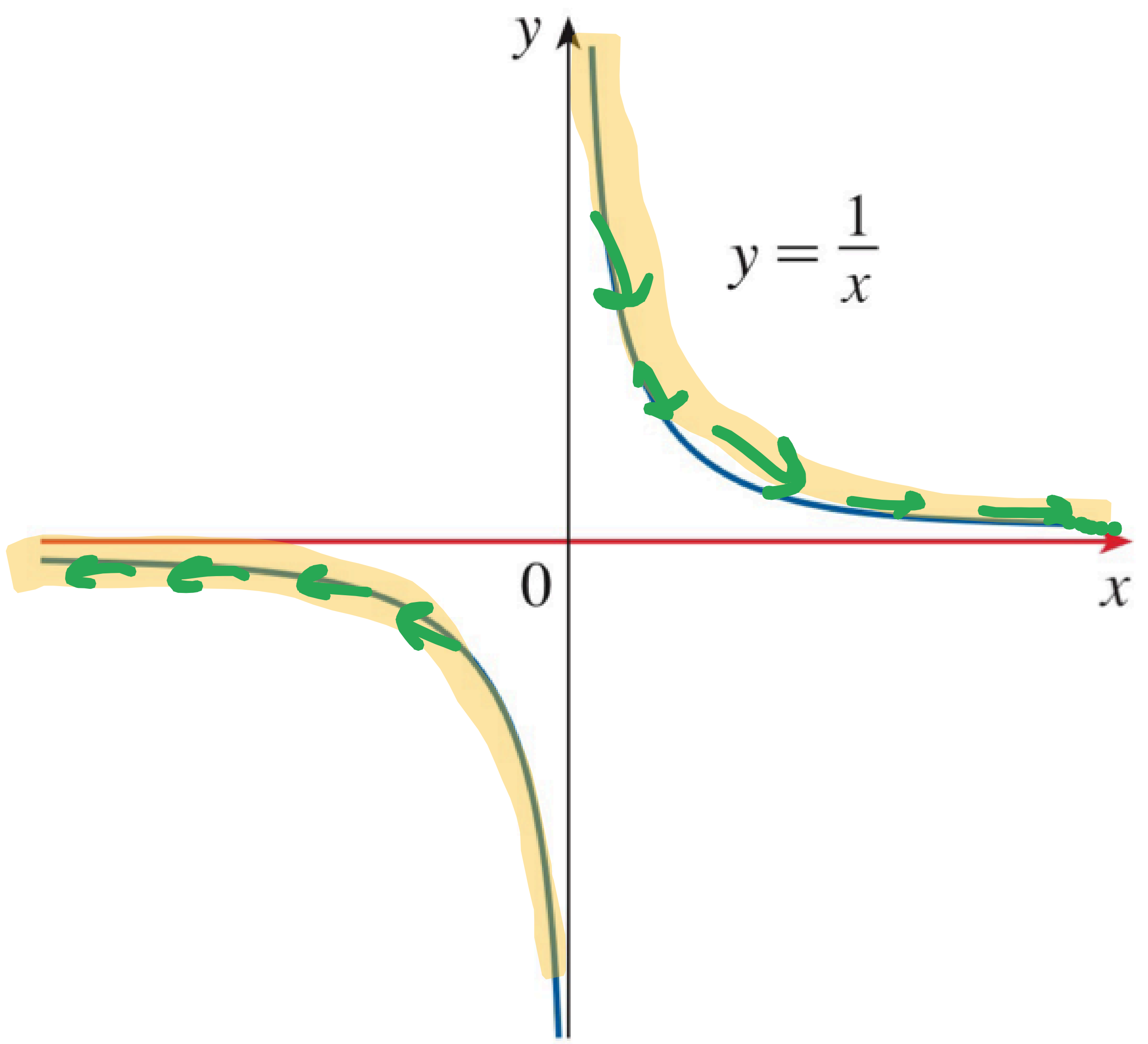
$\lim_{x \to \infty} f(x) = 3$

$y = 3$ is the horizontal asymptote

Find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$.

the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve

$y = 1/x$.



$$\lim_{x \to \infty} \frac{1}{x^2} = 0, \quad \lim_{x \to -\infty} \frac{1}{x} = 0$$

$$\frac{1}{\infty} = 0$$

$\boxed{5}$ **Theorem** If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Infinite Limits at Infinity

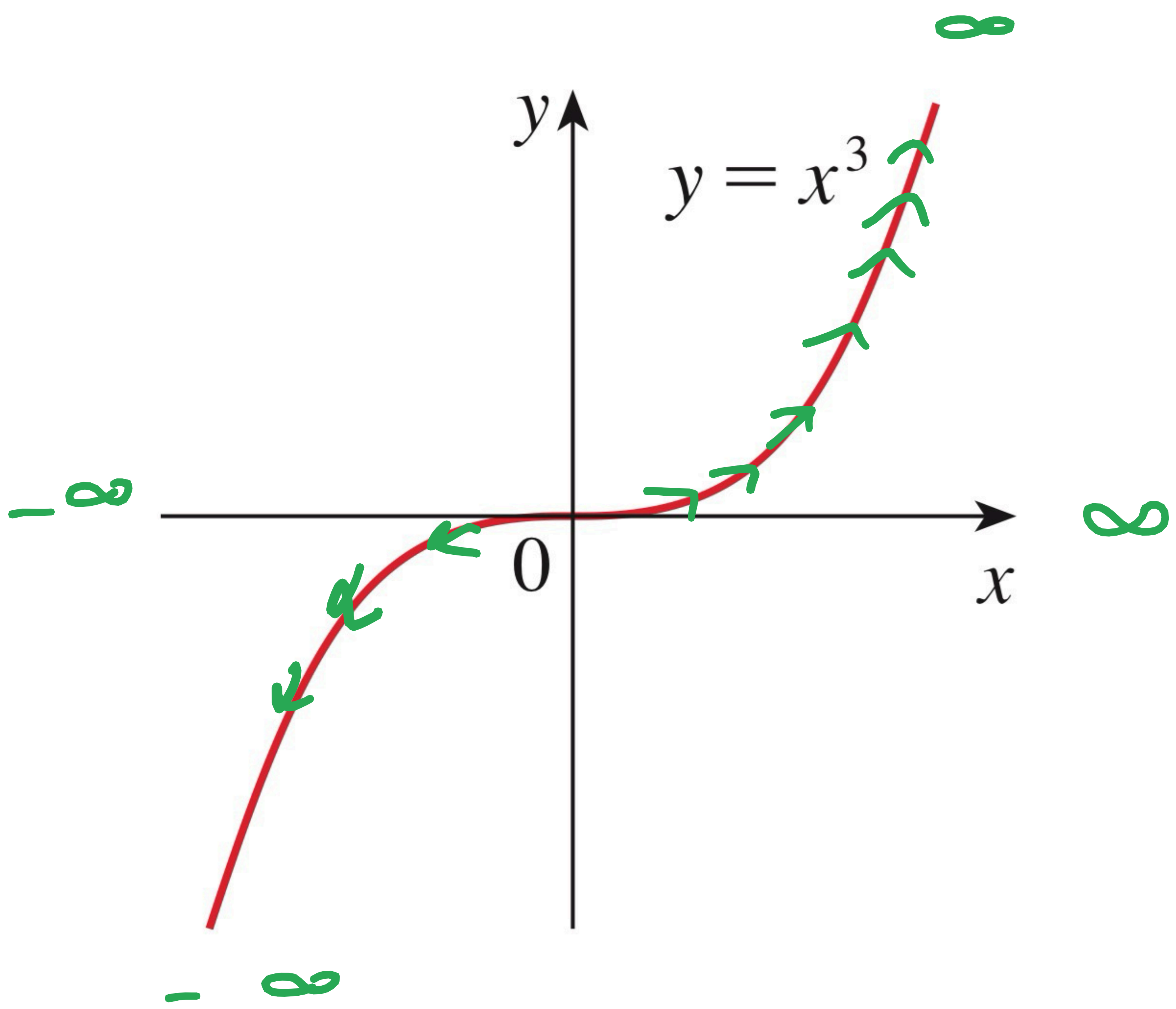
$$\lim_{x \to \infty} f(x) = \infty$$

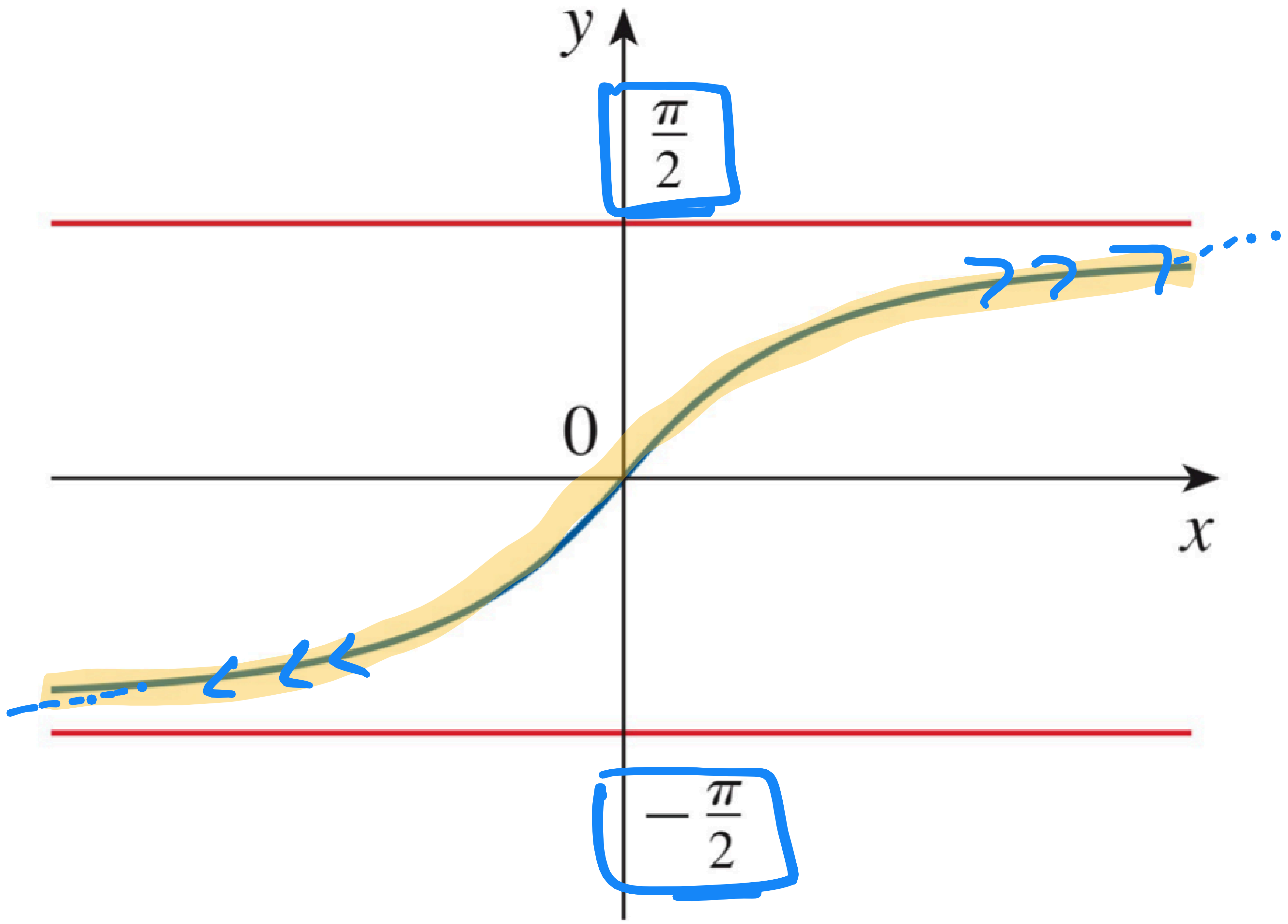
$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

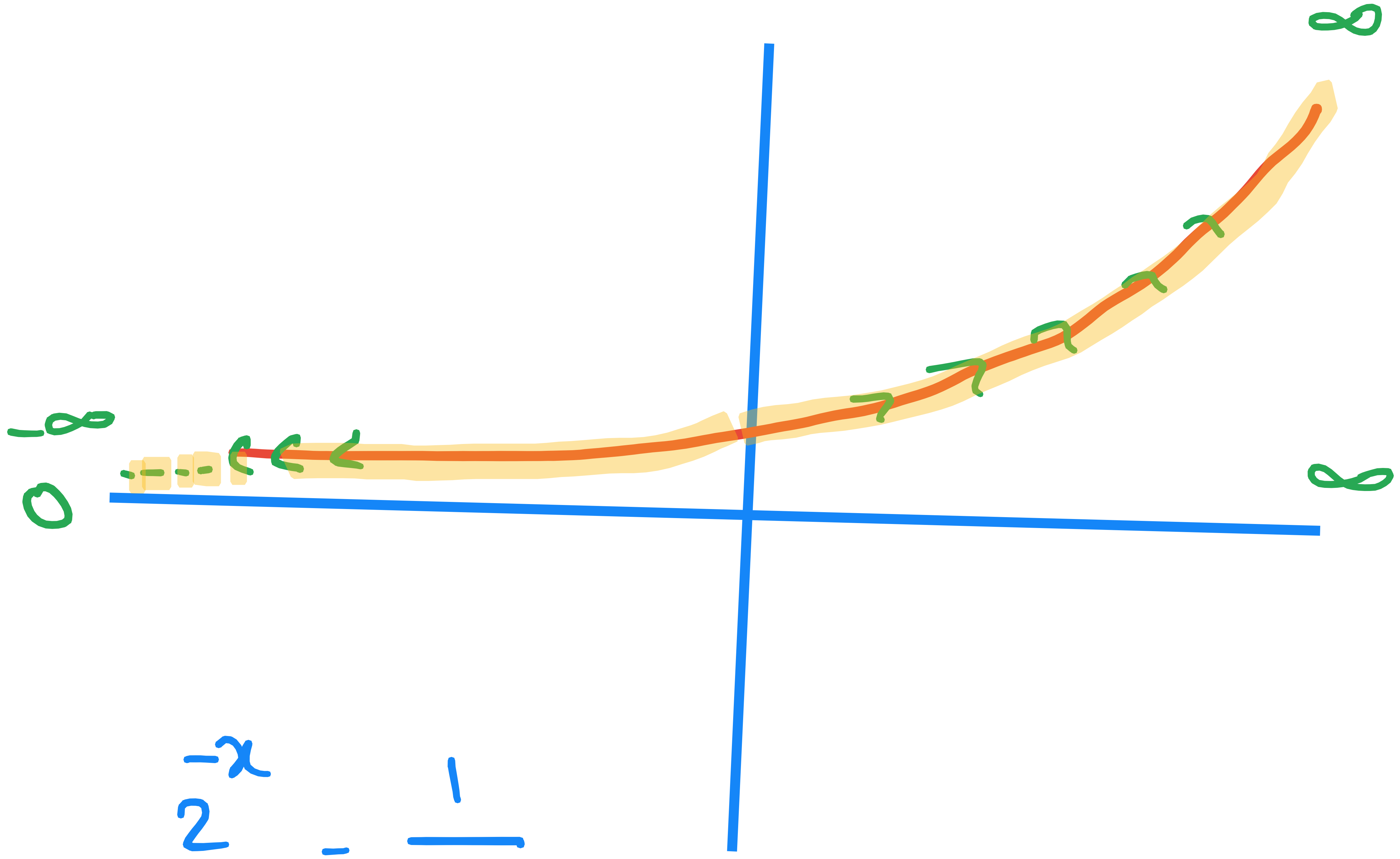
Find $\lim_{x \to \infty} x^3$ and $\lim_{x \to -\infty} x^3$.

$$\lim_{x \to \infty} x^3 = \infty \qquad (\infty)^3 = \infty$$

$$\lim_{x \to -\infty} x^3 = -\infty \qquad (-\infty)^2 = \infty$$







$$2^{-x} = \frac{1}{2^{x}}$$

$$\boxed{\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2} \qquad \lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}}$$

$$\boxed{\lim_{x \to \infty} e^{x} = \infty}$$

$$\lim_{x \to -\infty} e^{x} = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

Find $\lim\limits_{x \to \infty} (x^2 - x)$.

$\cancel{(\infty - \infty) = 0}$

$$\lim_{x \to \infty} x^2 = \infty \quad \text{and} \quad \lim_{x \to \infty} x = \infty$$

In general, the Limit Laws can't be applied to infinite limits because ∞ is not a number ($\infty - \infty$ can't be defined). However, we *can* write

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x(x - 1) = \infty$$

$\infty \cdot \infty = \infty$

$$\lim_{x \to \infty} x(x-1) = \infty$$

15–42 Find the limit or show that it does not exist.

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$$\lim_{x\to\pm\infty} \frac{1}{x} = 0$$

15. $\displaystyle \lim_{x\to\infty} \frac{4x+3}{5x-1}$

$$\lim_{x\to\infty} \frac{(4x+3)/x}{(5x-1)/x} = \lim_{x\to\infty} \frac{4+\frac{3}{x}}{5-\frac{1}{x}} = \frac{4+\cancel{\frac{3}{\infty}}\,0}{5-\cancel{\frac{1}{\infty}}\,0} = \frac{4}{5}$$

17. $\displaystyle \lim_{t\to-\infty} \frac{(3t^2+t)/t^3}{(t^3-4t+1)/t^3}$

$$\lim_{t\to-\infty} \frac{3/t+1/t^2}{1-4/t^2+1/t^3} = \frac{\cancel{\lim_{t\to-\infty}\frac{3}{t}}\,0+\cancel{\lim_{t\to-\infty}\frac{1}{t^2}}\,0}{\lim_{t\to-\infty}1-\cancel{\lim_{t\to-\infty}\frac{4}{t^2}}\,0+\cancel{\lim_{t\to-\infty}\frac{1}{t^3}}\,0} = \frac{0}{1} = 0$$

21. $\lim\limits_{x\to\infty} \dfrac{4-\sqrt{x}}{2+\sqrt{x}}$

$(\sqrt{x}) \to x^{1/2}$

$$\lim_{x\to\pm\infty} \frac{1}{x^r} = 0 \qquad r>0$$

$$\lim_{x\to\infty} \left(\frac{4-x^{1/2}}{2+x^{1/2}}\right) / x^{1/2} \to \lim_{x\to\infty} \frac{\frac{4}{\cancel{x^{1/2}}}^{0} - 1}{\frac{2}{\cancel{x^{1/2}}}^{0} + 1} = \frac{0-1}{0+1} = \frac{-1}{1} = -1$$

26. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{1+4x^6}\,)x^3}{(2-x^3)/x^3}$

$$\left(\left(\frac{1}{x^6}+4\right)x^6\right)^{1/2} / x^3$$

$$\left(\left(\frac{1}{x^6}+4\right)^{1/2} \cdot x^{\frac{6}{2}}\right) / x^3$$

$$\left(\left(\frac{1}{x^6}+4\right)^{1/2} \cdot \cancel{x^3}\right) / \cancel{x^3}$$

$$\lim_{x\to-\infty} \frac{\sqrt{1+4x^6} \,/\, x^3}{2/x^3 - 1} = \lim_{x\to-\infty}$$

$$\lim_{x\to-\infty} \frac{\sqrt{1/6+4}\; x^{\frac{6}{2}} \cdot \frac{1}{x^3}}{2/x^3 - 1}$$

26. $\lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$ $\to$ $\lim_{x\to-\infty} \frac{(1+4x^6)^{1/2}/x^3}{(2-x^3)/x^3}$

$$\lim_{x\to-\infty} \frac{(1+4x^6)^{1/2}/x^3}{2/x^3-1} \Rightarrow \lim_{x\to-\infty} \frac{((1/x^6+4)x^6)^{1/2}/x^3}{2/x^3-1}$$

$$\lim_{x\to-\infty} \frac{\left((1/x^6+4)^{1/2}\cdot x^{\boxed{\frac{6}{2}}}\right)/x^3}{2/x^3-1} \Rightarrow \lim_{x\to-\infty} \frac{\left(\sqrt{1/x^6+4}\;\cancel{x^3}\right)/\cancel{x^3}}{2/x^3-1}$$

$$\lim_{x\to-\infty} \frac{\sqrt{\underbrace{1/x^6}_{0}+4}}{\underbrace{2/x^3}_{0}-1} = \frac{\sqrt{4}}{-1} = \frac{2}{-1} = -2$$

$$\frac{\sqrt{x^3+1}}{x}$$

$$\cancel{\frac{\sqrt{x^3}}{x} + \frac{\sqrt{1}}{x}}$$

$$\frac{\sqrt{x^3}}{x} \rightarrow \sqrt{x^3/x} \rightarrow \sqrt{x^2} = x$$

27. $\lim_{x \to -\infty} \left(\frac{2x^5 - x}{x^4 + 3} \right) / x^4 \rightarrow \lim_{x \to -\infty} \frac{2x^5/x^4 - x/x^4}{x^4/x^4 + 3/x^4}$

$$\lim_{x \to -\infty} \frac{2x - \cancel{\frac{1}{x^3}}^{0}}{1 + \cancel{\frac{3}{x^4}}^{0}} \rightarrow \lim_{x \to -\infty} 2x = -\infty$$

29. $\lim_{t \to \infty} \left(\sqrt{25t^2 + 2} - 5t \right)$

29. $\lim_{t\to\infty}\left(\underbrace{\sqrt{25t^2+2}}\, - 5t\right) \rightarrow \cancel{\infty - \infty}$

$(25t^2+2)^{\boxed{2}}$

$\frac{1}{2}$

$\frac{1}{3}$

$$\lim_{t\to\infty} \sqrt{25t^2+2} - 5t \cdot \frac{\sqrt{25t^2+2}+5t}{\sqrt{25t^2+2}+5t}$$

$$\lim_{t\to\infty} \frac{(25t^2+2)-(5t)^2}{\sqrt{25t^2+2}+5t} = \lim_{t\to\infty} \frac{25\cancel{t^2}+2-\cancel{25t^2}}{\sqrt{25t^2+2}+5t}$$

$$\lim_{t\to\infty} \left(\frac{2}{\sqrt{25t^2+2}+5t}\right)/t \rightarrow \lim_{t\to\infty} \frac{\cancel{2/t}\ 0}{\left(\sqrt{25t^2+2}\right)/t+5} = \frac{0}{\sqrt{25}+5} = 0$$

$(25 + 2/t^2)^{1/2}(\cancel{t^2})^{1/2} \rightarrow \cancel{t/t}\ 1$

33. $\lim_{x \to -\infty} (x^2 + 2x^7)$ $\quad (-\infty)^2 + 2(-\infty)^7 \to \infty - \infty$

$$\lim_{x \to -\infty} x^7 \left(\frac{x^2}{x^7} + \frac{2\cancel{x^7}}{\cancel{x^7}} \right) \to \lim_{x \to -\infty} x^7 \left(\frac{1}{\cancel{x^5}} + 2 \right)$$

$$\lim_{x \to -\infty} x^7 (2) = -\infty$$

$$\boxed{\lim_{x \to \infty} e^x = \infty}$$

37. $\lim_{x \to \infty} \left(\dfrac{1 - e^x}{1 + 2e^x} \right) / e^x \to \lim_{x \to \infty} \dfrac{\frac{1}{\infty\ \boxed{e^x}} - \frac{\cancel{e^x}}{\cancel{e^x}}\ 1}{\frac{1}{\infty\ \boxed{e^x}} + 2\frac{\cancel{e^x}}{\cancel{e^x}}\ 2}$

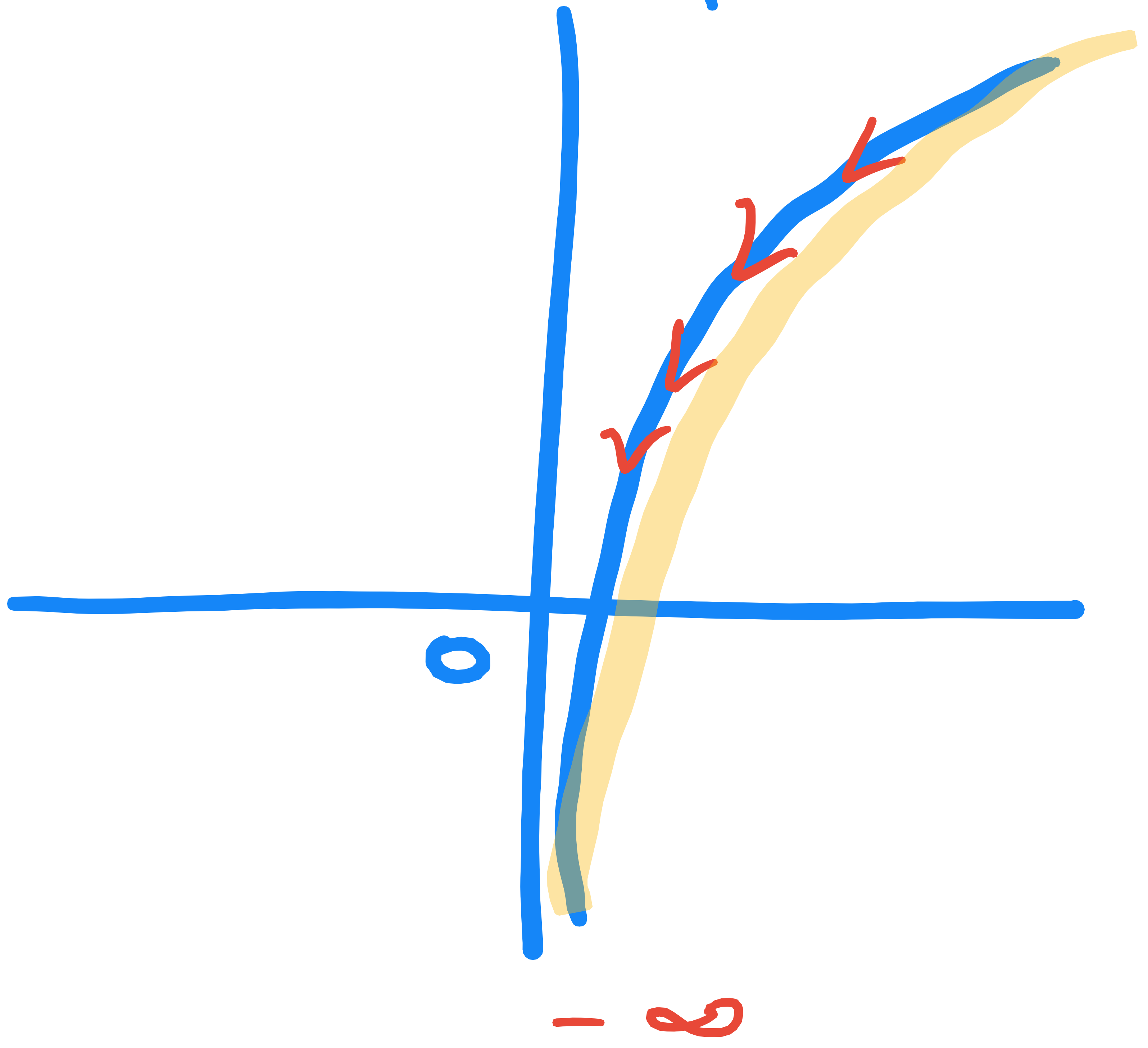
$$\frac{\cancel{\frac{1}{\infty}}\ 0 - 1}{\cancel{\frac{1}{\infty}}\ 0 + 2} = -\frac{1}{2}$$

40. $\lim_{x \to 0^+} \tan^{-1}(\ln x)$

$t = \ln x$

$\lim_{x \to 0^+} \ln(x) = -\infty$

$f(x) = \ln x$



$\lim_{t \to -\infty} \tan^{-1}(t) = -\frac{\pi}{2}$

42. $\lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$

$\lim_{x \to \infty} \ln\left(\frac{2+x}{1+x}\right) \to \lim_{x \to \infty} \ln\left(\frac{\frac{2}{x} + \frac{x}{x}}{\frac{1}{x} + \frac{x}{x}}\right)$

$\lim_{x \to \infty} \ln\left(\frac{\cancel{\frac{2}{x}}^{0} + 1}{\cancel{\frac{1}{x}}^{0} + 1}\right) \to \boxed{\ln(1) = 0}$