OL Academy

MATHS101

Lesson 4

2.5 Continuity

2.5 Continuity

$\lim_{x \to -4} f(x) = 1$

$f(-4) \Rightarrow$ not defined

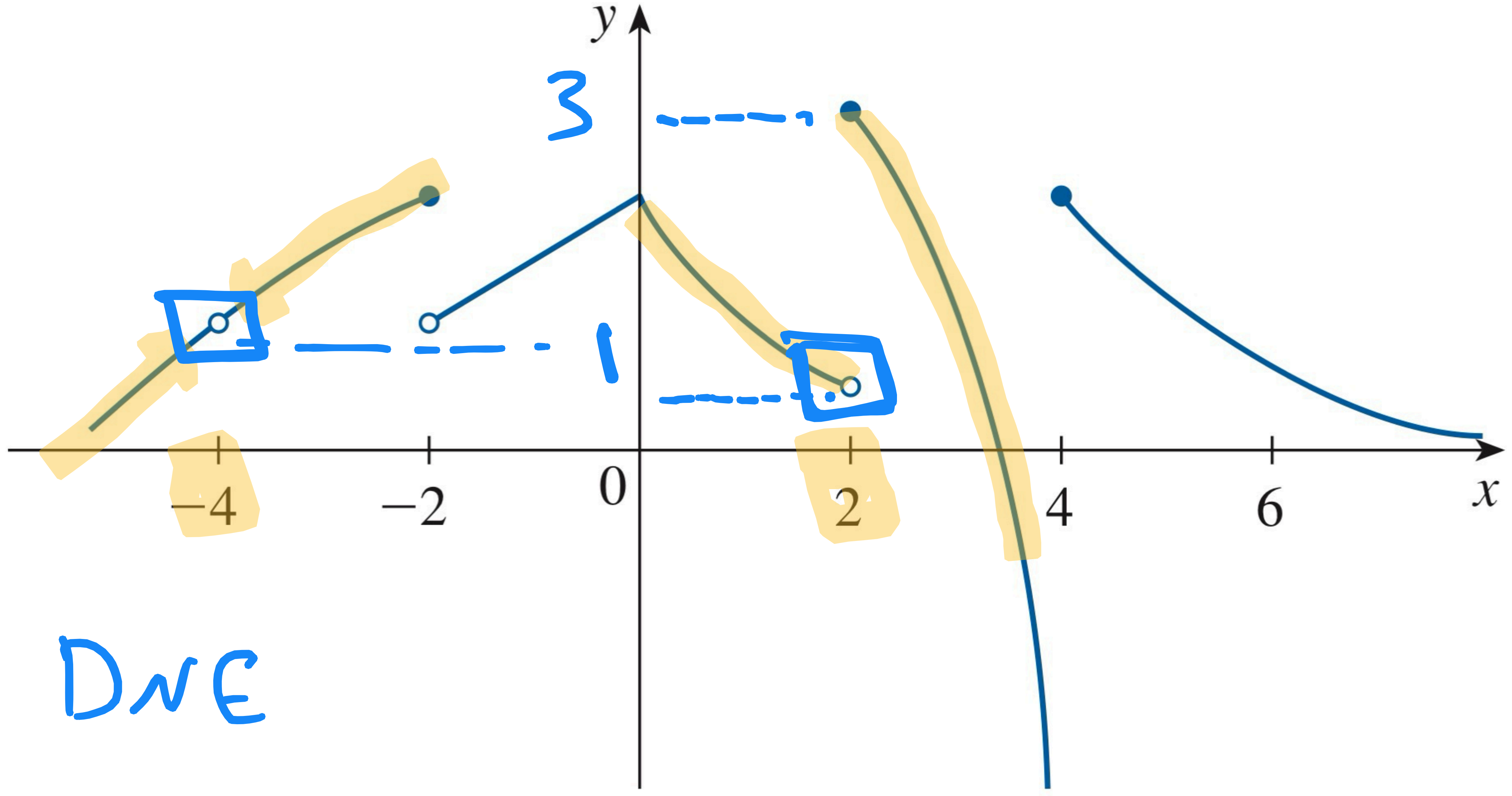
1 Definition A function f is **continuous at a number a** if

$$\lim_{x \to a} f(x) = f(a)$$

1. $f(a)$ is defined (that is, a is in the domain of f)

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$



$\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$ DNE

$f(2) \Rightarrow$ undefined

Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ $\rightarrow x - 2 = 0 \rightarrow x = 2$

$f(2)$ is undefined, so f is discontinuous at 2

(b) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ \boxed{1} & \underline{\text{if } x = 2} \end{cases}$

$\lim_{x \to 2} \underline{f(x)} = \underline{1} \Leftrightarrow f(2) = 1 \neq \lim_{x \to 2} f(x) = 3$

$x = 2$

$a = 1$, $b = -1$, $c = -2$

$x = -1$

$\lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2} = \dfrac{0}{0}$

$\lim_{x \to 2} \dfrac{(\cancel{x-2})(x+1)}{(\cancel{x-2})} \Rightarrow \lim_{x \to 2} x + 1 \Rightarrow 3$

13–16 Use the definition of continuity and the properties of limits to show that the function is continuous

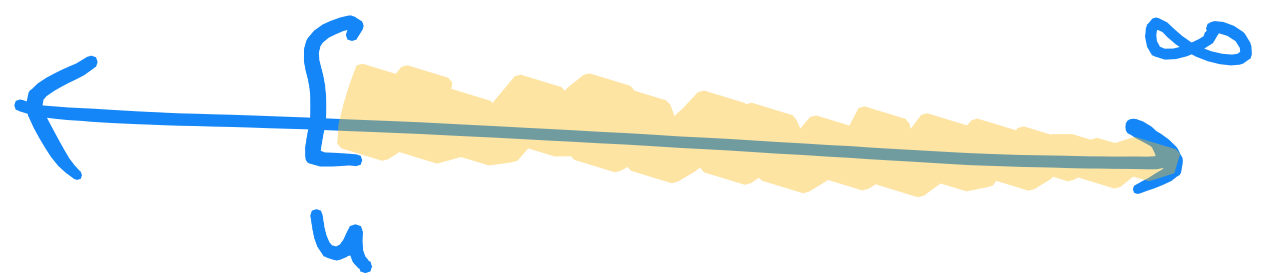
14. $g(t) = \dfrac{t^2 + 5t}{2t + 1}, \quad a = 2$

$$\Rightarrow \lim_{t \to 2} \frac{t^2 + 5t}{2t + 1} \Rightarrow \frac{\lim\limits_{t \to 2} (t^2 + 5t)}{\lim\limits_{t \to 2} (2t + 1)}$$

$$\frac{\lim\limits_{t \to 2} t^2 + \lim\limits_{t \to 2} 5t}{\lim\limits_{t \to 2} 2t + \lim\limits_{t \to 2} 1} \Rightarrow \frac{2^2 + 5(2)}{2(2) + 1} = \frac{14}{5}$$

$$g(2) = \frac{14}{5} = \lim_{t \to 2} g(t)$$

17. $f(x) = x + \sqrt{x - 4}\,, \quad [4, \infty)$ $\qquad a \ge 4$



$x \to 4$

$x \to 4^+$

$$\lim_{x \to 4} x + \sqrt{x - 4} = 4 + \sqrt{4 - 4} = 4$$

$$\lim_{x \to 4^+} x + \sqrt{x - 4} = 4$$

$$f(4) = 4 = \lim_{x \to 4} f(x)$$

19–24 Explain why the function is discontinuous at the given number a.

22. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} \rightarrow f(1) = 1$ $\qquad a = 1$

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} \Rightarrow \lim_{x \to 1} \frac{x^2 - x}{(x+1)(x-1)} \qquad a^2 - b^2 \Rightarrow (a+b)(a-b)$$

$$\lim_{x \to 1} \frac{x(\cancel{x-1})}{(x+1)(\cancel{x-1})} \Rightarrow \lim_{x \to 1} \boxed{\frac{x}{x+1}} \Rightarrow \frac{1}{1+1} = \frac{1}{2}$$

$$f(1) = 1, \quad \lim_{x \to 1} f(x) = \frac{1}{2} \Rightarrow 1 \neq \frac{1}{2}$$

23. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$ $\rightarrow 0^-$

$a = 0 \rightarrow f(0) = 0$

$\rightarrow 0^+$

$\lim_{x \to 0} \cos x = 1 \quad , \quad \lim_{x \to 0} 1 - x^2 = 1$

$f(0) = 0 \neq \lim_{x \to 0} f(x)$

(a) Show that f has a removable discontinuity at $x = 3$.
(b) Redefine $f(3)$ so that f is continuous at $x = 3$ (and thus the discontinuity is "removed").

26. $f(x) = \dfrac{x^2 - 7x + 12}{\boxed{x - 3}}$ $= 0$ $x = 3$

f(3) is undefined at $x=3$, so f is discontinuous at $x=3$

$a = 1$, $b = -7$, $c = 12$ mode 5 3 → $x = 3$, $x = 4$

$$f(x) = \frac{\cancel{(x-3)}(x-4)}{\cancel{x-3}} = x - 4$$

$\lim_{x \to 3} f(x) = x - 4 = -1$ → f has removable discontinuity at $x = 3$

47. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2^- \\ x^3 - cx & \text{if } x \geqslant 2^+ \end{cases} \begin{matrix} \to (-\infty, 2) \\ \to (2, \infty) \end{matrix}$$

$$\lim_{x \to 2^-} cx^2 + 2x = 4c + 4$$

$$\lim_{x \to 2^+} x^3 - cx = 8 - 2c$$

$$\Rightarrow 4c + 4 = 8 - 2c$$

$$4c + 2c = 8 - 4$$

$$6c = 4$$

$$c = \frac{4}{6} \to \boxed{c = \frac{2}{3}}$$

f continuous on $(-\infty, \infty)$, $c = \frac{2}{3}$

48. Find the values of a and b that make f continuous everywhere.

$(-\infty, \infty)$

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leqslant x < 3 \\ 2x - a + b & \text{if } x \geqslant 3 \end{cases}$$

at $x = 2$:

$$\lim_{x \to 2^-} f(x) \Rightarrow \lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^-} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}} = \lim_{x \to 2^-} x + 2 = 4$$

$$\lim_{x \to 2^+} f(x) \Rightarrow \lim_{x \to 2^+} ax^2 - bx + 3 = 4a - 2b + 3$$

$$4 = 4a - 2b + 3 \Rightarrow 4a - 2b = 1$$

48. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leqslant x < 3^- \\ 2x - a + b & \text{if } x \geqslant 3^+ \end{cases}$$

at $x = 3$

$$\lim_{x \to 3^-} f(x) \Rightarrow \lim_{x \to 3^-} ax^2 - bx + 3 = 9a - 3b + 3$$

$$\lim_{x \to 3^+} f(x) \Rightarrow \lim_{x \to 3^+} 2x - a + b = 6 - a + b$$

$$9a - 3b + 3 = 6 - a + b$$

$$\Rightarrow 10a - 4b = 3 \quad (2)$$

$$① \quad 4a - 2b = 1 \quad \times (-2) \Longrightarrow -8a + 4b = -2$$

$$② \quad 10a - 4b = 3$$

$$\begin{array}{r} -8a + 4b = -2 \\ 10a - 4b = 3 \quad + \\ \hline 2a = 1 \end{array} \qquad a = \frac{1}{2}$$

$$4\left(\frac{1}{2}\right) - 2b = 1 \Longrightarrow 2 - 2b = 1$$

$$-2b = 1 - 2 \Longrightarrow b = \frac{-1}{-2} = \frac{1}{2}$$

$a = b = \frac{1}{2}$, f continuous on $(-\infty, \infty)$

49. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \to 2} [3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

$$\lim_{x \to 2} [3f(x) + f(x)g(x)] = 36$$

$$3f(2) + 6f(2) = 36$$

$$9f(2) = 36$$

$$f(2) = \frac{36}{9} = 4$$

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Where are the following functions continuous? $h(x) = (f \circ g)(x)$

(a) $h(x) = \sin(x^2)$

$f(x) = \sin x$, $g(x) = x^2$

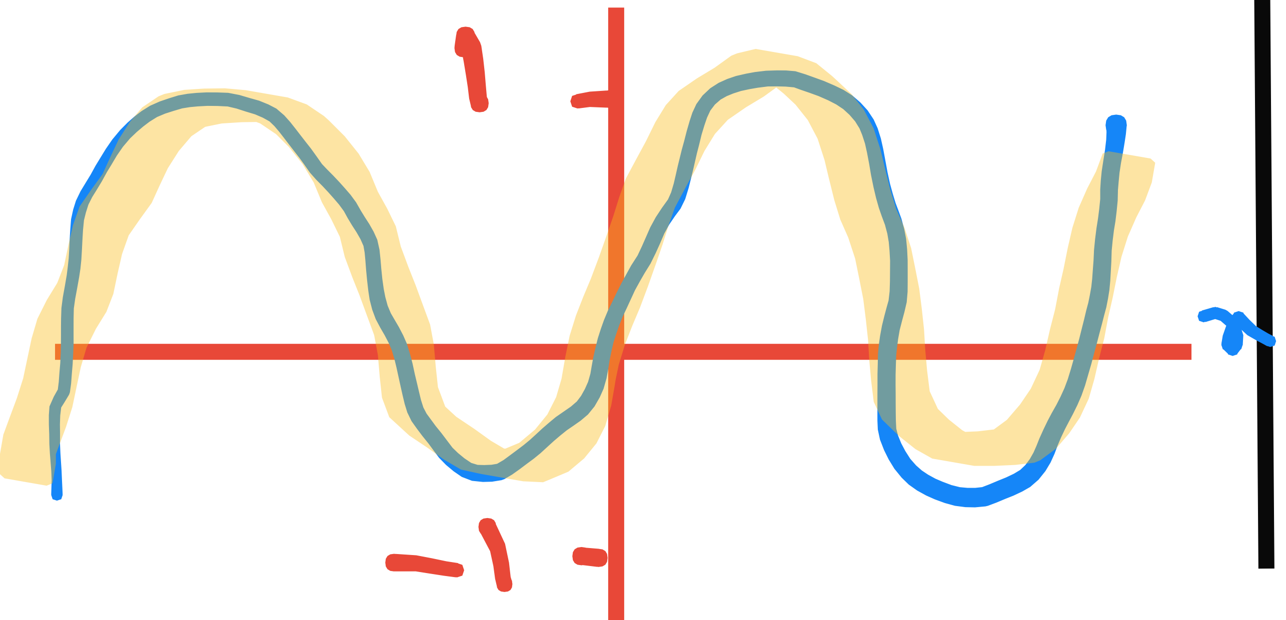
$\mathbb{R}$

Domain $(-\infty, \infty) \to x^2 \leftarrow$ Range $(0, \infty)$

Domain $(-\infty, \infty) \to \sin x \leftarrow$ Range $(-1, 1)$

Range $\to (0, 1)$

Continuous $(-\infty, \infty)$



$\lim_{x \to 2} x^2 = (2)^2 = 4$

$f(x) = \boxed{x^2}$

$g(x) = \ln(x)$

$\lim_{x \to 2} f(g(x))$

$\lim_{x \to 2} (\ln(x))^2$

$(f \circ g)(x) = f(g(x))$

$(g \circ f)(x) = g(f(x))$

(b) $F(x) = \underline{\ln(1 + \cos x)}$ $F(x) = h(g(x)) \rightarrow (h \circ g)(x)$

50. Let $f(x) = 1/x$ and $g(x) = 1/x^2$.

(a) Find $(f \circ g)(x)$. $\longrightarrow$ $f(g(x))$

(b) Is $f \circ g$ continuous everywhere? Explain.

$f(x) = \frac{1}{x}$, $g(x) = \frac{1}{x^2}$ $\rightarrow$ $f(\underbrace{g(x)}_{x}) = \frac{1}{x \rightarrow 1/x^2} = \frac{1}{1/x^2}$

$= x^2$

$f(x) = \frac{1}{x}$ $x \neq 0$, $g(x) = \frac{1}{x^2}$ $x \neq 0$

fog is continuous at $(-\infty, 0) \cup (0, \infty)$

fog is continuous everywhere except at $x = 0$