OL Academy

MATHS102

Lesson 2

4.4 L'Hopitals Rule

(2) Indeterminate Products (Type $0 \cdot \underline{\infty}$):

$\underline{\infty} \to \pm\infty$

$$\lim_{x \to a} \boxed{f(x) \cdot g(x)}$$

$f(x) \to 0$, $g(x) \to \mp\infty$

$$\frac{f(x)}{\frac{1}{g(x)}} \quad , \quad \frac{g(x)}{\frac{1}{f(x)}}$$

$$0 \cdot \infty \longrightarrow \frac{0}{0} \text{ or } \frac{\pm\infty}{\infty}$$

$\Rightarrow$ l'Hospitals Rule

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Examples:

(1) $\displaystyle \lim_{x \to 0^+} x^2 \ln(x)$ $\quad x^2 \to 0^2 = 0$, $\quad \ln(x) \to -\infty$ $\quad \to 0 \cdot -\infty$

$$= \lim_{x \to 0^+} \frac{\ln(x)}{x^{-2}} \to \frac{-\infty}{\infty}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-2x^{-3}}$$

$$= \lim_{x \to 0^+} \frac{x^3}{-2x}$$

$$= \lim_{x \to 0^+} \frac{x^2}{-2} = \frac{0^2}{-2} = \frac{0}{2} = 0$$

$$\frac{0}{n} = 0 \quad , n \neq 0$$

(2) $\lim_{x\to \pi/2^-} \tan x \cdot \ln(\sin x)$

$\tan x \to \infty$, $\ln(\sin x) \to 0$: $\infty \cdot 0$

$$= \lim_{x\to \pi/2^-} \frac{\ln(\sin x)}{\cot x} \to \frac{0}{0}$$

$$= \lim_{x\to \pi/2^-} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x}$$

$$= \lim_{x\to \pi/2^-} -\frac{\cos x}{\cancel{\sin x}} \cdot \sin^{\cancel{2}\,1} x$$

$$= \lim_{x\to \pi/2^-} -\cos x \sin x$$

$$= -(0)(1) = 0$$

(3) $\lim_{x\to\infty} x^3 e^{-x^2}$

$x^3 \to \infty^3 = \infty$, $e^{-x^2} \to e^{-\infty^2} = e^{-\infty} = 0$

$$= \lim_{x\to\infty} \frac{x^3}{e^{x^2}} \to \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{3x^{\cancel{2}\,1}}{2\cancel{x}e^{x^2}}$$

$$= \lim_{x\to\infty} \frac{3x}{2e^{x^2}} \to \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{3}{2\cdot e^{x^2}\cdot 2x}$$

$$= \lim_{x\to\infty} \frac{3}{4xe^{x^2}} = \frac{3}{4(\infty)(\infty)}$$

$$= \frac{3}{\infty} = 0$$

(3) Indeterminate Differences $(\infty - \infty)$:

$$\lim_{x \to a} [f(x) - g(x)]$$

quotient

$$\frac{f(x)}{g(x)}$$

$$\frac{0}{0} - \frac{\Delta}{\Delta}$$

$$\Rightarrow \frac{\square}{\square}$$

$$\frac{0}{0}, \quad \frac{\infty}{\infty}$$

product

$$f(x) \cdot g(x)$$

عامل مشترك

$$0 \cdot \infty$$

$$\rightarrow \frac{0}{0}, \quad \frac{\infty}{\infty}$$

Examples:

(1) $$\lim_{x \to 0^+} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)$$ $\infty - \infty$

$$= \lim_{x \to 0^+} \frac{x - (e^x - 1)}{x(e^x - 1)} \rightarrow \frac{0}{0}$$

$$= \lim_{x \to 0^+} \frac{1 - e^x}{(e^x - 1) + x e^x} \rightarrow \frac{0}{0}$$

$$= \lim_{x \to 0^+} \frac{-e^x}{e^x + e^x + x e^x}$$

$$= \frac{-e^0}{e^0 + e^0 + 0} = \frac{-1}{1+1} = -\frac{1}{2}$$

(2) $\lim_{x \to 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x} \right)$

(1) Indeterminate Powers (0^0, ∞^0, 1^∞):

$$\lim_{x\to a} [f(x)]^{g(x)} \quad y = \square$$

① $\ln(y) = \ln[f(x)]^{g(x)}$

$= g(x)\ln(f(x))$

② $\lim_{x\to a} [\ln(y)] = L$

③ $\lim_{x\to a} [f(x)]^{g(x)} = e^L$

Examples:

(1) $\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} \cancel{= 1} \Rightarrow 1^\infty$

$\to \frac{1}{0} = \infty$

$1+x = 1+0 = 1$

set $y = \ln((1+x)^{\frac{1}{x}})$

$= \frac{1}{x}\ln(1+x)$

$\lim_{x\to 0^+} \frac{1}{x}\ln(1+x) \to 0\cdot\infty$

$= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \to \frac{0}{0}$

$= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = \frac{\frac{1}{1+0}}{1} = 1$

$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e^1 = e$

(2) $\lim\limits_{x \to 1^-} (1-x)^{\ln(x)}$

(3) $\lim\limits_{x \to \infty} x^{\frac{1}{x}}$

Exercises:

(1) $\lim\limits_{x \to 0} \dfrac{e^{ax} - 1}{\sin x}$, $a =$ constant. $\rightarrow \dfrac{0}{0}$

$$= \lim_{x \to 0} \frac{a \cdot e^{ax}}{\cos x}$$

$$= \frac{a \cdot e^{0}}{\cos(0)} = \frac{a \cdot (1)}{(1)} = a$$

---

(2) If $\lim\limits_{x \to 0} \dfrac{\sin(ax) + bx^3 - 2x}{x^3} = 0$. Find a and b.

$$\lim_{x \to 0} \frac{\sin(ax) + bx^3 - 2x}{\cancel{3x^2}\; x^3} \rightarrow \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{a \cdot \cos(ax) + 3bx^2 - 2}{3\cancel{6}x^2} = 0 \qquad \rightarrow \frac{0}{0}$$

$\rightarrow 0$

$a\cos(0) + 0 - 2 = 0$

$a - 2 = 0 \Longrightarrow \boxed{a = 2}$

$$= \lim_{x \to 0} \frac{2\cos(2x) + 3bx^2 - 2}{3\cancel{6}x^2}$$

$$= \lim_{x \to 0} \frac{-4\sin(2x) + 6bx}{6x} \begin{matrix} \rightarrow 0 \\ \\ \rightarrow 0 \end{matrix} \quad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\boxed{-8\cos(2x) + 6b}}{6} = \frac{0}{6} = 0$$

$-8\cos(2x) + 6b = 0$

$-8\cancel{\cos}(0) + 6b = 0$

$-8 + 6b = 0 \Longrightarrow \boxed{b = \dfrac{8}{6} = \dfrac{4}{3}}$

(3) $\lim_{x \to \infty} \left(\frac{2x-3}{2x+5} \right)^{2x+1}$