OL Academy

MATHS102

Calculus II

Lesson (1)
(4.4) L'Hopital's Rule

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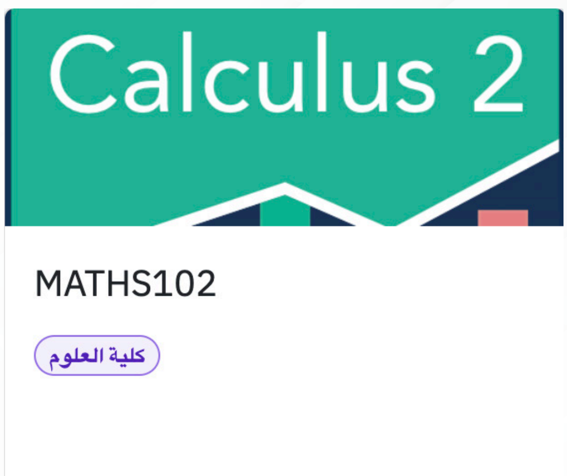
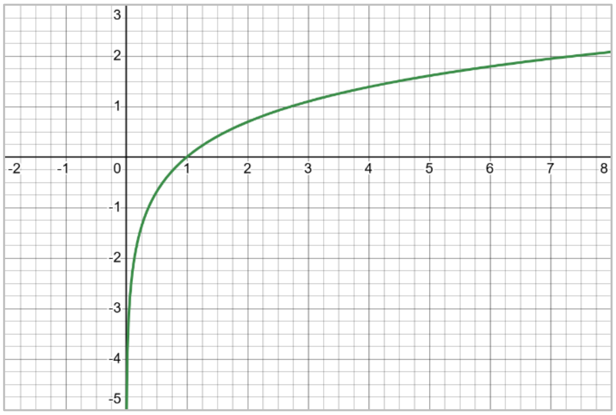


4.4 – L'Hospital's Rule

Notice that,

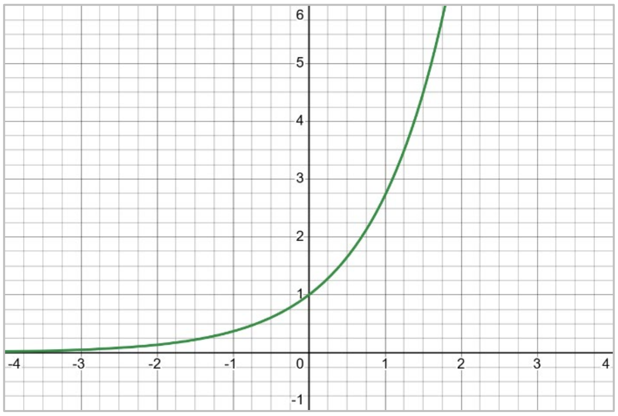


1. $f(x) = ln(x)$

   Domain: $(0, \infty)$

   $\lim_{x \to \infty} ln(x) = \infty$

   $\lim_{x \to 0^+} ln(x) = -\infty$

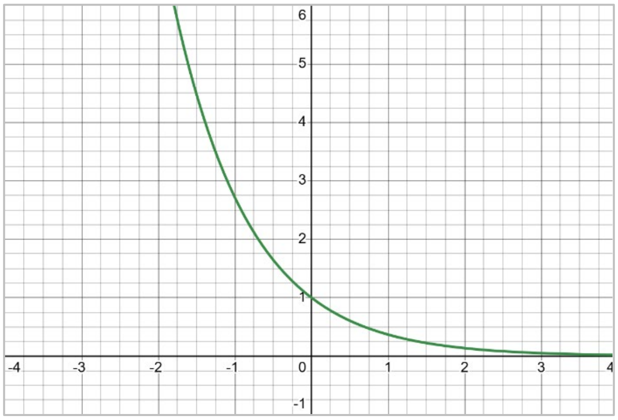


2. $f(x) = e^x$

   Domain: $(-\infty, \infty)$

   $\lim_{x \to \infty} e^x = \infty$

   $\lim_{x \to -\infty} e^x = 0$

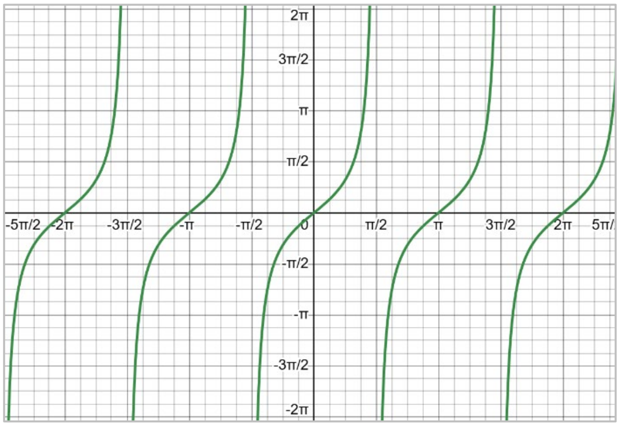


3. $f(x) = e^{-x}$

   Domain: $(-\infty, \infty)$

   $\lim_{x \to \infty} e^{-x} = 0$

   $\lim_{x \to -\infty} e^{-x} = \infty$

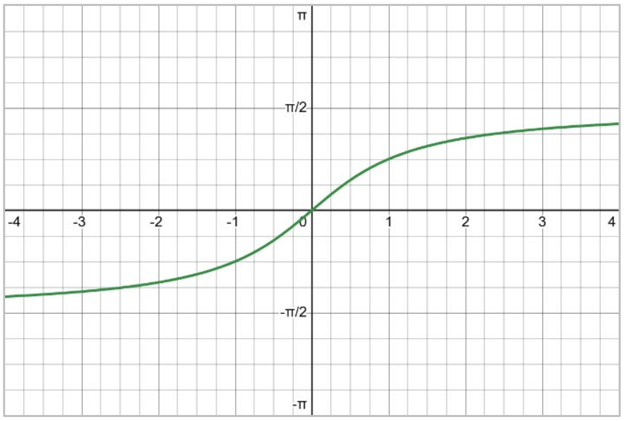


4. $f(x) = tan\, x$

   Domain: $\left\{x | x \neq \frac{(2n+1)\pi}{2}\right\}$

   $\lim_{x \to \pm\frac{\pi}{2}^+} \tan x = -\infty$

   $\lim_{x \to \pm\frac{\pi}{2}^-} \tan x = \infty$



5. $f(x) = tan^{-1} x$

Domain: $(-\infty, \infty)$

$$\lim_{x\to\infty} tan^{-1} x = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} tan^{-1} x = -\frac{\pi}{2}$$

Sometimes it is difficult to evaluate a limit because of indeterminate forms.

In this section, we will discuss these forms and how to evaluate the limit for each one.

(1) Indeterminate Forms (Types $\frac{0}{0}, \frac{\infty}{\infty}$):

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is $\frac{0}{0}$ or $\frac{\infty}{\infty}$, where a can be a real number or ∞, then we can apply l'Hospital's Rule.

- **L'Hospital's Rule:**

If we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

➢ **Examples:**

1. $\lim_{x\to 0} \frac{\sin x}{x} \Rightarrow$ Type $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{\cos x}{1} = \cos(0) = 1$$

$\frac{d}{dx}(\sin x) = \cos x$

2. $\lim_{x\to 1} \frac{x^2-2x+1}{x^2-1} \Rightarrow$ Type $\frac{0}{0}$

$$= \lim_{x\to 1} \frac{2x-2}{2x} = \frac{2(1)-2}{2(1)} = \frac{0}{2} = 0$$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(c) = 0$, c is constant

3. $\lim_{x\to\infty} \frac{e^x}{x^2} \Rightarrow$ Type $\frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \frac{e^\infty}{2} = \frac{\infty}{2} = \infty$$

"Notice that we can apply l'Hospital's Rule more than once until we get an answer"

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(c) = 0$, c is constant

4. $\lim_{x\to\frac{\pi}{2}^-} \frac{\sec x}{1+\tan x} \Rightarrow$ Type $\frac{\infty}{\infty}$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\sin x}{\cos x} \cdot \cos x$$

$$= \lim_{x\to\frac{\pi}{2}^-} \sin x = \sin\left(\frac{\pi}{2}\right) = 1$$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\tan x = \frac{\sin x}{\cos x}$, $\frac{1}{\sec x} = \cos x$

5. $\lim_{x\to\infty} \frac{e^{-x}}{\pi/2-\tan^{-1}x} \Rightarrow$ Type $\frac{0}{0}$

$= \lim_{x\to\infty} \frac{-e^{-x}}{\frac{-1}{1+x^2}} = \lim_{x\to\infty} \frac{1+x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(e^{-x}) = -e^{-x}$

6. $\lim_{x\to\infty} \frac{(\ln x)^2}{x} \Rightarrow$ Type $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{2\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2}{x} = \frac{2}{\infty} = 0$

$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x)$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

7. $\lim_{x\to 0} \frac{e^x - e^{-x}}{2\sin(2x)} \Rightarrow$ Type $\frac{0}{0}$

$= \lim_{x\to 0} \frac{e^x + e^{-x}}{4\cos(2x)} = \frac{e^0+e^0}{4\cos(2(0))} = \frac{1+1}{4(1)} = \frac{2}{4} = \frac{1}{2}$

$\frac{d}{dx}(\sin f(x)) = f'(x)\cos f(x)$

$\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$

8. $\lim_{x\to 2} \frac{\sqrt{x+2}-2}{x^2-4} \Rightarrow$ Type $\frac{0}{0}$

$= \lim_{x\to 2} \frac{1/2\sqrt{x+2}}{2x} = \lim_{x\to 2} \frac{1}{4x\sqrt{x+2}} = \frac{1}{4(2)\sqrt{2+2}} = \frac{1}{16}$

$\frac{d}{dx}(\sqrt{f(x)}) = \frac{f'(x)}{2f(x)}$

$\frac{d}{dx}(x^n) = nx^{n-1}$

9. $\lim_{x\to 0} \frac{x\cdot 3^x}{3^x-1} \Rightarrow$ Type $\frac{0}{0}$

$= \lim_{x\to 0} \frac{3^x + x\cdot 3^x\cdot \ln(3)}{3^x\cdot \ln(3)} = \lim_{x\to 0} \frac{3^x(1+x\ln(3))}{3^x\cdot \ln(3)} = \lim_{x\to 0} \frac{1+x\ln(3)}{\ln(3)} = \frac{1}{\ln(3)}$

$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

$\frac{d}{dx}(b^x) = b^x \cdot \ln b,\ b > 0$

(2) Indeterminate Products (Type $0\cdot\infty$):

If $\lim_{x\to a} f(x)\cdot g(x)$ is $0\cdot\infty$, then write $f(x)\cdot g(x)$ in the form $\frac{f(x)}{\frac{1}{g(x)}}$ or $\frac{g(x)}{\frac{1}{f(x)}}$ to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and apply l'Hospital's Rule.

➢ **Examples:**

1. $\lim_{x\to 0^+} x^2\ln x \Rightarrow$ Type $0\cdot\infty$

$= \lim_{x\to 0^+} \frac{\ln x}{x^{-2}} = \lim_{x\to 0^+} \frac{x^{-1}}{-2x^{-3}} = \lim_{x\to 0^+} \frac{x^3}{-2x} = \lim_{x\to 0^+} \frac{x^2}{-2} = \frac{0}{-2} = 0$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

2. $\lim_{x\to\frac{\pi}{2}^-} \tan x \ln(\sin x) \Rightarrow$ Type $0\cdot\infty$

$= \lim_{x\to\frac{\pi}{2}^-} \frac{\ln(\sin x)}{\cot x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\cos x/\sin x}{-\csc^2 x} = \lim_{x\to\frac{\pi}{2}^-} -\frac{\cos x}{\sin x}\cdot \sin^2 x$

$= \lim_{x\to\frac{\pi}{2}^-} -\cos x \sin x = 0$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$

3. $\lim_{x\to\infty} x^3 e^{-x^2}$ ⇒ Type $0\cdot\infty$

$$= \lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{3x^2}{2xe^{x^2}} = \lim_{x\to\infty} \frac{3x}{2e^{x^2}} = \lim_{x\to\infty} \frac{3}{4xe^{x^2}} = \frac{3}{\infty} = 0$$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$

(3) Indeterminate Differences (Type $\infty - \infty$):

If $\lim_{x\to a} f(x) - g(x)$ is $\infty - \infty$, then try to change $f(x) - g(x)$ to a product or a quotient to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and apply l'Hospital's Rule.

➢ **Examples:**

1. $\lim_{x\to 0^+} \left(\frac{1}{e^x-1} - \frac{1}{x}\right)$ ⇒ Type $\infty - \infty$

$$= \lim_{x\to 0^+} \frac{x-e^x+1}{x(e^x-1)} = \lim_{x\to 0^+} \frac{1-e^x}{e^x+xe^x-1} = \lim_{x\to 0^+} \frac{-e^x}{e^x+e^x+xe^x}$$

$$= \lim_{x\to 0^+} \frac{-e^x}{2e^x+xe^x}$$

$$= \lim_{x\to 0^+} \frac{-e^x}{e^x(2+x)} = \lim_{x\to 0^+} \frac{-1}{2+x} = -\frac{1}{2}$$

$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{A}{B} - \frac{C}{D} = \frac{AD-BC}{BD}$

2. $\lim_{x\to 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x}\right)$ ⇒ Type $\infty - \infty$

$$= \lim_{x\to 0^+} \frac{(3x+1)\sin x - x}{x\sin x} = \lim_{x\to 0^+} \frac{3\sin x + (3x+1)\cos x - 1}{\sin x + x\cos x}$$

$$= \lim_{x\to 0^+} \frac{3\cos x + 3\cos x - (3x+1)\sin x}{\cos x + \cos x - x\sin x} = \frac{6}{2} = 3$$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{A}{B} - \frac{C}{D} = \frac{AD-BC}{BD}$

(4) Indeterminate Powers (Types $0^0, \infty^0, 1^\infty$):

If $\lim_{x\to a} [f(x)]^{g(x)}$ is 0^0 or ∞^0 or 1^∞ , then set $y = \ln[f(x)]^{g(x)} = g(x)\ln f(x)$ and evaluate $\lim_{x\to a} y$ to get $0\cdot\infty$.

If $\lim_{x\to a} y = L$, then $\lim_{x\to a} [f(x)]^{g(x)} = e^L$.

Examples:

1. $\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} \Rightarrow$ Type 1^{∞}

Set $y = \ln(1+x)^{\frac{1}{x}} = \frac{1}{x}\ln(1+x)$

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+}\frac{1}{x}\ln(1+x) = \lim_{x\to 0^+}\frac{\ln(1+x)}{x} = \lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1}$$
$$= \lim_{x\to 0^+}\frac{1}{1+x} = \frac{1}{1} = 1$$

$\to \lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = e^1 = e$

$\ln[f(x)]^{g(x)} = g(x)\ln[f(x)]$

$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$

$\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\lim_{x\to 1^-}(1-x)^{\ln x} \Rightarrow$ Type 0^0

Set $y = \ln(1-x)^{\ln x} = \ln x\ln(1-x)$

$$\lim_{x\to 1^-} y = \lim_{x\to 1^-}(\ln x\ln(1-x)) = \lim_{x\to 1^-}\frac{\ln(1-x)}{\frac{1}{\ln x}}$$
$$= \lim_{x\to 1^-}\frac{-\frac{1}{1-x}}{\frac{1}{x(\ln x)^2}}$$
$$= \lim_{x\to 1^-}\frac{x(\ln x)^2}{1-x}$$
$$= \lim_{x\to 1^-}\frac{(\ln x)^2+2\ln x}{-1} = 0$$

$\to \lim_{x\to 1^-}(1-x)^{\ln x} = e^0 = 1$

$\ln[f(x)]^{g(x)} = g(x)\ln[f(x)]$

$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$

$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$

$\frac{d}{dx}(x^n) = nx^{n-1}$

3. $\lim_{x\to\infty}(x)^{\frac{1}{x}} \Rightarrow$ Type ∞^0

Set $y = \ln(x)^{\frac{1}{x}} = \frac{1}{x}\ln x$

$$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(\frac{1}{x}\ln x\right) = \lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0$$

$\lim_{x\to\infty}(x)^{\frac{1}{x}} = e^0 = 1$

$\ln[f(x)]^{g(x)} = g(x)\ln[f(x)]$

$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$

$\frac{d}{dx}(x^n) = nx^{n-1}$

4.4 – L'Hospital's Rule – Exercises

❖ **Evaluate the following limits:**

1. $\lim\limits_{x\to 3} \frac{x-3}{x^2-9}$ → Type $\frac{0}{0}$

$\lim\limits_{x\to 3}(x-3) = 3-3 = 0$, $\lim\limits_{x\to 3}(x^2-9) = 3^3 - 9 = 0$

$= \lim\limits_{x\to 3} \frac{1}{2x} = \frac{1}{2(3)} = \frac{1}{6}$

2. $\lim\limits_{x\to 0} \frac{e^{2x}-1}{\sin x}$ → Type $\frac{0}{0}$

$\lim\limits_{x\to 0} e^{2x} - 1 = e^0 - 1 = 0$, $\lim\limits_{x\to 0} \sin x = \sin(0) = 0$

$= \lim\limits_{x\to 0} \frac{2e^{2x}}{\cos x} = \frac{2e^0}{\cos(0)} = \frac{2(1)}{1} = 2$

$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$

$\frac{d}{dx}(\sin x) = \cos x$

3. $\lim\limits_{x\to 1} \frac{\ln x}{\sin(\pi x)}$ → Type $\frac{0}{0}$

$\lim\limits_{x\to 1} \ln x = \ln(1) = 0$, $\lim\limits_{x\to 1} \sin(\pi x) = \sin(\pi) = 0$

$= \lim\limits_{x\to 1} \frac{\frac{1}{x}}{\pi\cos(\pi x)} = \lim\limits_{x\to 1} \frac{1}{\pi x \cos(\pi x)} = \frac{1}{\pi(1)\cos(\pi)} = -\frac{1}{\pi}$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\frac{d}{dx}\left(\sin(f(x))\right) = f'(x)\cos(f(x))$

4. $\lim\limits_{x\to 0} \frac{x-\sin x}{x-\tan x}$ → Type $\frac{0}{0}$

$\lim\limits_{x\to 0}(x - \sin x) = 0 - \sin(0) = 0$, $\lim\limits_{x\to 0}(x - \tan x) = 0 - \tan(0) = 0$

$= \lim\limits_{x\to 0} \frac{1-\cos x}{1-\sec^2 x}$ → $\frac{0}{0}$

$= \lim\limits_{x\to 0} \frac{-(-\sin x)}{-2\sec x \sec x \tan x}$

$= \lim\limits_{x\to 0} \frac{\sin x}{-2\sec^2 x \frac{\sin x}{\cos x}} = \lim\limits_{x\to 0} \frac{\sin x \cos x}{-2\sec^2 x \sin x} = \lim\limits_{x\to 0} \frac{\cos x}{-2\sec^2 x} = \frac{\cos(0)}{-2\sec^2(0)} = -\frac{1}{2}$

$\frac{d}{dx}\left([f(x)]^n\right) = n[f(x)]^{n-1}f'(x)$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

5. $\lim\limits_{x\to 0^+} \frac{\tan^{-1}(2x)}{\ln x}$

$= \frac{\tan^{-1}(0)}{\ln(0)} = \frac{0}{-\infty} = 0$

6. $\lim\limits_{x\to 0^+} \sin x \ln x$ → Type $0 \cdot \infty$

$\lim\limits_{x\to 0^+} \sin x = \sin(0) = 0$, $\lim\limits_{x\to 0^+} \ln x = -\infty$

$= \lim\limits_{x\to 0^+} \frac{\ln x}{\csc x}$ → $\frac{\infty}{\infty}$

$= \lim\limits_{x\to 0^+} \frac{1/x}{-\csc x \cot x}$

$= \lim\limits_{x\to 0^+} \frac{-\sin x \tan x}{x}$ → $\frac{0}{0}$

$= \lim\limits_{x\to 0^+} \frac{-(\cos x \tan x + \sin x \sec^2 x)}{1}$

$= -\cos(0)\tan(0) - \sin(0)\sec^2(0) = 0$

$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\csc x = \frac{1}{\sin x}$, $\cot x = \frac{1}{\tan x}$

7. $\lim_{x\to1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$ → Type $\infty-\infty$

$\lim_{x\to1}\frac{x}{x-1}=\frac{1}{1-1}=\frac{1}{0}=\infty$, $\lim_{x\to1}\frac{1}{\ln x}=\frac{1}{\ln(1)}=\frac{1}{0}=\infty$

$=\lim_{x\to1}\frac{x\ln x-(x-1)}{(x-1)\ln x}\to\frac{0}{0}$

$=\lim_{x\to1}\frac{\ln x+x\cdot\frac{1}{x}-1}{\ln x+\frac{x-1}{x}}$

$=\lim_{x\to1}\frac{\ln x}{\ln x+\frac{x-1}{x}}\to\frac{0}{0}$

$=\lim_{x\to1}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{(1)(x)-(x-1)(1)}{x^2}}=\lim_{x\to1}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}}=\frac{1}{1+1}=\frac{1}{2}$

$\left(\frac{f(x)}{g(x)}\right)'=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

$(f(x)g(x))'=f'(x)g(x)+f(x)g'(x)$

$\frac{d}{dx}(\ln x)=\frac{1}{x}$

$\frac{A}{B}-\frac{C}{D}=\frac{AD-BC}{BD}$

8. $\lim_{x\to0^+}x^{\sqrt{x}}$ → Type 0^0

$\lim_{x\to0^+}x=0$, $\lim_{x\to0^+}\sqrt{x}=\sqrt{0}=0$

Set $y=\ln(x)^{\sqrt{x}}=\sqrt{x}\ln x$

$\lim_{x\to0^+}\sqrt{x}\ln x\to0\cdot\infty$

$=\lim_{x\to0^+}\frac{\ln x}{x^{-\frac{1}{2}}}\to\frac{\infty}{\infty}$

$=\lim_{x\to0^+}\frac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}}=\lim_{x\to0^+}\frac{-2x^{\frac{3}{2}}}{x}=\lim_{x\to0^+}-2x^{\frac{1}{2}}=-2(0)=0$

So, $\lim_{x\to0^+}x^{\sqrt{x}}=e^0=1$

$\ln[f(x)]^{g(x)}=g(x)\ln[f(x)]$

$\frac{d}{dx}(\ln x)=\frac{1}{x}$

$\sqrt[n]{x^m}=x^{\frac{m}{n}}$

$\frac{x^a}{x^b}=x^{a-b}$

$\frac{A/B}{C/D}=\frac{AD}{BC}$

$x^{-a}=\frac{1}{x^a}$

9. $\lim_{x\to0^+}(4x+1)^{\cot x}$ → Type 1^∞

$\lim_{x\to0^+}(4x+1)=4(0)+1=0$, $\lim_{x\to0^+}\cot x=\cot(0)=\frac{\cos(0)}{\sin(0)}=\frac{1}{0}=\infty$

Set $y=\ln(4x+1)^{\cot x}=\cot x\ln(4x+1)$

$\lim_{x\to0^+}\cot x\ln(4x+1)\to0\cdot\infty$

$=\lim_{x\to0^+}\frac{\ln(4x+1)}{\tan x}\to\frac{0}{0}$

$=\lim_{x\to0^+}\frac{\frac{4}{4x+1}}{\sec^2 x}=\lim_{x\to0^+}\frac{4}{(4x+1)\sec^2 x}=\lim_{x\to0^+}\frac{4\cos^2 x}{4x+1}=\frac{4\cos^2(0)}{4(0)+1}=4$

So, $\lim_{x\to0^+}(4x+1)^{\cot x}=e^4$

$\ln[f(x)]^{g(x)}=g(x)\ln[f(x)]$

$\frac{d}{dx}(\tan x)=\sec^2 x$

$\frac{d}{dx}(\ln[f(x)])=\frac{f'(x)}{f(x)}$

$\sec x=\frac{1}{\cos x}$, $\cot x=\frac{\cos x}{\sin x}$

10. $\lim_{x\to1}\frac{x-1}{\ln x-\sin(\pi x)}$ → Type $\frac{0}{0}$

$\lim_{x\to1}(x-1)=1-1=0$, $\lim_{x\to1}(\ln x-\sin(\pi x))=\ln(1)-\sin(\pi)=0$

$=\lim_{x\to1}\frac{1}{\frac{1}{x}-\pi\cos(\pi x)}=\frac{1}{1-\pi\cos(\pi)}=\frac{1}{1-(\pi)(-1)}=\frac{1}{1+\pi}$

$\frac{d}{dx}(\ln x)=\frac{1}{x}$

$\frac{d}{dx}(\sin(f(x)))=f'(x)\cos(f(x))$

11. $\lim\limits_{x\to\frac{\pi}{2}} \frac{\ln(\csc x)}{\left(x-\frac{\pi}{2}\right)^2}$ → Type $\frac{0}{0}$

$\lim\limits_{x\to\frac{\pi}{2}} \ln(\csc x) = \lim\limits_{x\to\frac{\pi}{2}} \ln\left(\frac{1}{\sin x}\right)$, $\lim\limits_{x\to\frac{\pi}{2}} \left(x-\frac{\pi}{2}\right)^2 = \left(\frac{\pi}{2}-\frac{\pi}{2}\right)^2 = 0$

$= \lim\limits_{x\to\frac{\pi}{2}} [\ln(1) - \ln(\sin x)]$

$= \lim\limits_{x\to\frac{\pi}{2}} [-\ln(\sin x)] = -\ln\left(\sin\left(\frac{\pi}{2}\right)\right) = -\ln(1) = 0$

$= \lim\limits_{x\to\frac{\pi}{2}} \frac{\frac{-\csc x \cot x}{\csc x}}{2\left(x-\frac{\pi}{2}\right)(1)}$

$= \lim\limits_{x\to\frac{\pi}{2}} -\frac{\cot x}{2\left(x-\frac{\pi}{2}\right)}$ → $\frac{0}{0}$

$= \lim\limits_{x\to\frac{\pi}{2}} -\frac{-\csc^2 x}{2} = \lim\limits_{x\to\frac{\pi}{2}} \frac{\csc^2 x}{2} = \frac{\csc^2\left(\frac{\pi}{2}\right)}{2} = \frac{1}{2}\cdot\frac{1}{\sin^2\left(\frac{\pi}{2}\right)} = \frac{1}{2}$

$\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1}f'(x)$

$\ln\left(\frac{f(x)}{g(x)}\right) = \ln[f(x)] - \ln[g(x)]$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\ln[f(x)]) = \frac{f'(x)}{f(x)}$

$\csc x = \frac{1}{\sin x}$

12. $\lim\limits_{x\to\infty} x^2 e^{-x}$ → Type $0\cdot\infty$

$\lim\limits_{x\to\infty} x^2 = \infty^2 = \infty$, $\lim\limits_{x\to\infty} e^{-x} = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

$= \lim\limits_{x\to\infty} \frac{x^2}{e^x}$ → $\frac{\infty}{\infty}$

$= \lim\limits_{x\to\infty} \frac{2x}{e^x}$ → $\frac{\infty}{\infty}$

$= \lim\limits_{x\to\infty} \frac{2}{e^x} = \frac{2}{e^{\infty}} = \frac{2}{\infty} = 0$

$\frac{d}{dx}(e^x) = e^x$

$\frac{1}{e^x} = e^{-x}$

13. $\lim\limits_{x\to 0} \frac{(e^x-1)^2}{x\sin x}$ → Type $\frac{0}{0}$

$\lim\limits_{x\to 0}(e^x-1)^2 = \left(e^0-1\right)^2 = 0$, $\lim\limits_{x\to 0}(x\sin x) = (0)(\sin(0)) = 0$

$= \lim\limits_{x\to 0} \frac{2(e^x-1)(e^x)}{\sin x + x\cos x}$ → $\frac{0}{0}$

$= \lim\limits_{x\to 0} \frac{2(e^x)(e^x)+2(e^x)(e^x-1)}{\cos x+\cos x - x\sin x}$

$= \lim\limits_{x\to 0} \frac{2e^{2x}+2e^x(e^x-1)}{2\cos x + x\sin x} = \frac{2e^0+2e^0(e^0-1)}{2\cos(0)+(0)(\sin(0))} = \frac{2}{2} = 1$

$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$

$\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1}f'(x)$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(e^x) = e^x$

14. $\lim\limits_{x\to 0}(e^x+x)^{\frac{1}{x}}$ → Type 1^{∞}

$\lim\limits_{x\to 0}(e^x+x) = e^0+0 = 1$, $\lim\limits_{x\to 0}\frac{1}{x} = \frac{1}{0} = \infty$

Set $y = \ln(e^x+x)^{\frac{1}{x}} = \frac{1}{x}\ln(e^x+x)$

$\lim\limits_{x\to 0}\left[\frac{1}{x}\ln(e^x+x)\right]$ → $0\cdot\infty$

$= \lim\limits_{x\to 0}\frac{\ln(e^x+x)}{x}$ → $\frac{0}{0}$

$= \lim\limits_{x\to 0}\frac{\frac{e^x+1}{e^x+x}}{1} = \lim\limits_{x\to 0}\frac{e^x+1}{e^x+x} = \frac{e^0+1}{e^0+0} = \frac{1+1}{1} = 2$

So, $\lim\limits_{x\to 0}(e^x+x)^{\frac{1}{x}} = e^2$

$\ln[f(x)]^{g(x)} = g(x)\ln[f(x)]$

$\frac{d}{dx}(\ln[f(x)]) = \frac{f'(x)}{f(x)}$

$\frac{d}{dx}(e^x) = e^x$

15. $\lim\limits_{x\to 0^+} \ln x - \ln(\sin x)$ → Type $\infty - \infty$

$\lim\limits_{x\to 0^+} \ln x = -\infty$, $\lim\limits_{x\to 0^+} \ln(\sin x) = \infty$

$= \lim\limits_{x\to 0^+} \ln\left(\frac{x}{\sin x}\right) \to \ln\left(\frac{0}{0}\right)$

$= \lim\limits_{x\to 0^+} \left[-\ln\left(\frac{\sin x}{x}\right)\right] = -\ln(1) = 0$

$\lim\limits_{x\to 0^+} \frac{\sin x}{x} \to \frac{0}{0}$

$= \lim\limits_{x\to 0^+} \frac{\cos x}{1} = \cos(0) = 1$

$\ln[f(x)] - \ln[g(x)] = \ln\left(\frac{f(x)}{g(x)}\right)$

$\ln\left(\frac{f(x)}{g(x)}\right) = -\ln\left(\frac{g(x)}{f(x)}\right)$

$\frac{d}{dx}(\sin x) = \cos x$

16. $\lim\limits_{x\to \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right)\tan x$ → Type $0 \cdot \infty$

$\lim\limits_{x\to \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right) = \frac{\pi}{2} - \frac{\pi}{2} = 0$, $\lim\limits_{x\to \frac{\pi}{2}^-} \tan x = \frac{\sin\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\right)} = \frac{1}{0} = \infty$

$= \lim\limits_{x\to \frac{\pi}{2}^-} \frac{\left(\frac{\pi}{2} - x\right)}{\cot x} \to \frac{0}{0}$

$= \lim\limits_{x\to \frac{\pi}{2}^-} \frac{-1}{-\csc^2 x} = \lim\limits_{x\to \frac{\pi}{2}^-} \sin^2 x = \sin^2\left(\frac{\pi}{2}\right) = 1$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\sin x = \frac{1}{\csc x}$

$\cot x = \frac{1}{\tan x}$

17. $\lim\limits_{x\to 0} \frac{3^x - 1}{2^x - 1}$ → Type $\frac{0}{0}$

$\lim\limits_{x\to 0}(3^x - 1) = 3^0 - 1 = 1 - 1 = 0$, $\lim\limits_{x\to 0}(2^x - 1) = 2^0 - 1 = 1 - 1 = 0$

$= \lim\limits_{x\to 0} \frac{\ln(3)\cdot 3^x}{\ln(2)\cdot 2^x} = \frac{\ln(3)\cdot 3^0}{\ln(2)\cdot 2^0} = \frac{\ln(3)}{\ln(2)}$

$\frac{d}{dx}(b^x) = \ln(b)\cdot b^x$, $b > 0$

18. $\lim\limits_{x\to \infty} \frac{\log_2(x)}{\log_3(x+3)}$ → Type $\frac{\infty}{\infty}$

$\lim\limits_{x\to \infty} \log_2(x) = \lim\limits_{x\to \infty} \frac{\ln(x)}{\ln(2)} = \infty$, $\lim\limits_{x\to \infty} \log_3(x+3) = \lim\limits_{x\to \infty} \frac{\ln(x+3)}{\ln(3)} = \infty$

$= \lim\limits_{x\to \infty} \frac{\frac{1}{\ln(2)\cdot x}}{\frac{1}{\ln(3)\cdot(x+3)}}$

$= \lim\limits_{x\to \infty} \frac{\ln(3)}{\ln(2)} \cdot \frac{x+3}{x}$

$= \lim\limits_{x\to \infty} \frac{\ln(3)}{\ln(2)} \cdot \frac{1}{1} = \frac{\ln(3)}{\ln(2)}$

$\log_b[f(x)] = \frac{\ln(f(x))}{\ln(b)}$

$\frac{d}{dx}(\log_b[f(x)]) = \frac{f'(x)}{\ln(b)\cdot f(x)}$

$\frac{1/A}{1/B} = \frac{B}{A}$

19. $\lim\limits_{x\to 0} \frac{\sqrt{ax + a^2} - a}{x}$, where $a > 0$. → Type $\frac{0}{0}$

$\lim\limits_{x\to 0} \sqrt{ax + a^2} - a = \sqrt{a^2} - a = a - a = 0$, $\lim\limits_{x\to 0} x = 0$

$= \lim\limits_{x\to 0} \frac{\frac{a}{2\sqrt{ax + a^2}}}{1} = \lim\limits_{x\to 0} \frac{a}{2\sqrt{ax + a^2}} = \frac{a}{2\sqrt{a^2}} = \frac{a}{2a} = \frac{1}{2}$

$\sqrt[n]{x^m} = \left|x^{\frac{m}{n}}\right|$ if $\frac{m}{n}$ is odd number and n is even number

$\frac{d}{dx}\left(\sqrt{f(x)}\right) = \frac{f'(x)}{2\sqrt{f(x)}}$

20. $\lim_{x\to\infty}\left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}}$ → Type ∞^0

$\lim_{x\to\infty}\frac{x^2+1}{x+2}=\infty$, $\lim_{x\to\infty}\frac{1}{x}=\frac{1}{\infty}=0$

Set $y=\ln\left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}}=\frac{1}{x}\ln\left(\frac{x^2+1}{x+2}\right)$

$\lim_{x\to\infty}\left[\frac{1}{x}\ln\left(\frac{x^2+1}{x+2}\right)\right]$ → $0\cdot\infty$

$=\lim_{x\to\infty}\frac{\ln\left(\frac{x^2+1}{x+2}\right)}{x}$

$=\lim_{x\to\infty}\frac{\ln(x^2+1)-\ln(x+2)}{x}$ → $\frac{\infty}{\infty}$

$=\lim_{x\to\infty}\left[\frac{2x}{x^2+1}-\frac{1}{x+2}\right]=\lim_{x\to\infty}\left(\frac{2x}{x^2+1}\right)-\lim_{x\to\infty}\left(\frac{1}{x+2}\right)=0$

So, $\lim_{x\to\infty}\left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}}=e^0=1$

$\ln\left(\frac{f(x)}{g(x)}\right)=\ln[f(x)]-\ln[g(x)]$

$\ln[f(x)]^{g(x)}=g(x)\ln[f(x)]$

$\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)$

$\frac{d}{dx}(\ln[f(x)])=\frac{f'(x)}{f(x)}$

We have $f(x)=\frac{ax^m+\cdots}{bx^n+\cdots}$:

1. If $m>n$, then $\lim_{x\to\infty}f(x)=\pm\infty$

2. If $m=n$, then $\lim_{x\to\infty}f(x)=\frac{a}{b}$

3. If $m<n$, then $\lim_{x\to\infty}f(x)=0$

21. $\lim_{x\to\infty}e^x(1-\cos(e^{-x}))$ → Type $0\cdot\infty$

$\lim_{x\to\infty}e^x=e^\infty=\infty$, $\lim_{x\to\infty}(1-\cos(e^{-x}))=1-\cos(0)=0$

$=\lim_{x\to\infty}\frac{1-\cos(e^{-x})}{e^{-x}}$ → $\frac{0}{0}$

$=\lim_{x\to\infty}\frac{-\sin(e^{-x})\cdot e^{-x}}{-e^{-x}}$

$=\lim_{x\to\infty}\sin(e^{-x})=\sin(e^{-\infty})=\sin(0)=0$

$\frac{d}{dx}(\cos(f(x)))=-f'(x)\sin(f(x))$

$\frac{d}{dx}(e^{f(x)})=f'(x)e^{f(x)}$

22. $\lim_{x\to 0}\frac{\cos\left(\frac{\pi}{2}+x\right)-\sin x}{\tan x}$ → Type $\frac{0}{0}$

$\lim_{x\to 0}\left[\cos\left(\frac{\pi}{2}+x\right)-\sin x\right]=0$, $\lim_{x\to 0}\tan x=\tan(0)=0$

$=\lim_{x\to 0}\frac{\cos\left(\frac{\pi}{2}\right)\cos x-\sin\left(\frac{\pi}{2}\right)\sin x-\sin x}{\tan x}$

$=\lim_{x\to 0}\frac{-\sin x-\sin x}{\tan x}$

$=\lim_{x\to 0}\frac{-2\sin x}{\tan x}$

$=\lim_{x\to 0}\frac{-2\cos x}{\sec^2 x}=\lim_{x\to 0}-2(\cos x)(\cos^2 x)=\lim_{x\to 0}-2\cos^3 x=-2$

$\frac{d}{dx}(\sin x)=\cos x$

$\frac{d}{dx}(\tan x)=\sec^2 x$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

23. $\lim_{x\to 0}\frac{e^{ax}-1}{\sin x}$, where a is a constant → Type $\frac{0}{0}$

$\lim_{x\to 0}e^{ax}-1=e^0-1=0$, $\lim_{x\to 0}\sin x=\sin(0)=0$

$=\lim_{x\to 0}\frac{ae^{2x}}{\cos x}=\frac{ae^0}{\cos(0)}=\frac{a(1)}{1}=a$

$\frac{d}{dx}(e^{f(x)})=f'(x)e^{f(x)}$

$\frac{d}{dx}(\sin x)=\cos x$

24. $\lim\limits_{x\to\frac{\pi}{4}}(1-\tan x)\sec(2x)$ → Type $0\cdot\infty$

$\lim\limits_{x\to\frac{\pi}{4}}(1-\tan x)=1-\tan\left(\frac{\pi}{4}\right)=0\,,\ \lim\limits_{x\to\frac{\pi}{4}}\sec(2x)=\frac{1}{\cos\left(\frac{\pi}{2}\right)}=\frac{1}{0}=\infty$

$=\lim\limits_{x\to\frac{\pi}{4}}\frac{(1-\tan x)}{\cos(2x)}\to\frac{0}{0}$

$=\lim\limits_{x\to\frac{\pi}{4}}\frac{-\sec^2 x}{-2\sin(2x)}=\frac{\sec^2\left(\frac{\pi}{4}\right)}{2\sin\left(\frac{\pi}{2}\right)}=\frac{2}{2}=1$

$\frac{d}{dx}\left(\cos(f(x))\right)=-f'(x)\sin(f(x))$

$\frac{d}{dx}(\tan x)=\sec^2 x$

$\cos x=\frac{1}{\sec x}$

25. $\lim\limits_{x\to\infty}\frac{\ln(1+e^x)}{1+x}$ → Type $\frac{\infty}{\infty}$

$\lim\limits_{x\to\infty}\ln(1+e^x)=\ln(1+\infty)=\infty\,,\ \lim\limits_{x\to\infty}(1+x)=\infty$

$=\lim\limits_{x\to\infty}\frac{e^x}{1+e^x}\to\frac{\infty}{\infty}$

$=\lim\limits_{x\to\infty}\frac{e^x}{e^x}=\lim\limits_{x\to\infty}1=1$

$\frac{d}{dx}(e^x)=e^x$

$\frac{d}{dx}\left(\ln(f(x))\right)=\frac{f'(x)}{f(x)}$

26. $\lim\limits_{x\to1}\frac{\ln x}{x-1}$ → Type $\frac{0}{0}$

$\lim\limits_{x\to1}\ln x=\ln(1)=0\,,\ \lim\limits_{x\to\infty}(x-1)=1-1=0$

$=\lim\limits_{x\to1}\frac{1/x}{1}=\frac{1}{1}=1$

$\frac{d}{dx}(\ln x)=\frac{1}{x}$

27. $\lim\limits_{x\to0}\frac{e^x+e^{-x}-2}{e^x-x-1}$ → Type $\frac{0}{0}$

$\lim\limits_{x\to0}(e^x+e^{-x}-2)=1+1-2=0\,,\ \lim\limits_{x\to0}(e^x-x-1)=1-1=0$

$=\lim\limits_{x\to0}\frac{e^x-e^{-x}}{e^x-1}\to\frac{0}{0}$

$=\lim\limits_{x\to0}\frac{e^x+e^{-x}}{e^x}=\frac{1+1}{1}=2$

$\frac{d}{dx}(e^x)=e^x$

$\frac{d}{dx}\left(e^{f(x)}\right)=f'(x)e^{f(x)}$

28. $\lim\limits_{x\to0}\frac{\sin^{-1}x}{x}$ → Type $\frac{0}{0}$

$\lim\limits_{x\to0}\sin^{-1}x=\sin^{-1}(0)=0\,,\ \lim\limits_{x\to0}x=0$

$=\lim\limits_{x\to0}\frac{\frac{1}{\sqrt{1-x^2}}}{1}=\frac{1}{\sqrt{1}}=1$

$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$

29. $\lim\limits_{x\to0^+}\frac{\tan^{-1}(2x)}{\ln x}$

$=\frac{\tan^{-1}(0)}{\infty}=\frac{0}{\infty}=0$

30. $\lim\limits_{x\to1}\frac{x\sin(x-1)}{2x^2-x-1}$ → Type $\frac{0}{0}$

$\lim\limits_{x\to1}(x\sin(x-1))=\sin(0)=0\,,\ \lim\limits_{x\to1}\left(2x^2-x-1\right)=2-1-1=0$

$=\lim\limits_{x\to1}\frac{\sin(x-1)+x\cos(x-1)}{4x-1}=\frac{\sin(0)+\cos(0)}{4-1}=\frac{1}{3}$

$\left(f(x)g(x)\right)'=f'(x)g(x)+f(x)g'(x)$

$\frac{d}{dx}\left(\sin(f(x))\right)=f'(x)\cos(f(x))$

31. $\lim\limits_{x\to\infty}(e^x+10x)^{\frac{1}{x}}$ → Type ∞^0

$\lim\limits_{x\to\infty}(e^x+10x)=\infty$, $\lim\limits_{x\to\infty}\frac{1}{x}=\frac{1}{\infty}=0$

Set $y=\ln(e^x+10x)^{\frac{1}{x}}=\frac{1}{x}\ln(e^x+10x)$

$\lim\limits_{x\to\infty}\left[\frac{1}{x}\ln(e^x+10x)\right]$ → $0\cdot\infty$

$=\lim\limits_{x\to\infty}\frac{\ln(e^x+10x)}{x}$ → $\frac{\infty}{\infty}$

$=\lim\limits_{x\to\infty}\frac{e^x+10}{e^x+10x}$ → $\frac{\infty}{\infty}$

$=\lim\limits_{x\to\infty}\frac{e^x}{e^x+10}$ → $\frac{\infty}{\infty}$

$=\lim\limits_{x\to\infty}\frac{e^x}{e^x}=\lim\limits_{x\to\infty}1=1$

So, $\lim\limits_{x\to\infty}(e^x+10x)^{\frac{1}{x}}=e^1=e$

$\ln[f(x)]^{g(x)}=g(x)\ln[f(x)]$

$\frac{d}{dx}(\ln[f(x)])=\frac{f'(x)}{f(x)}$

$\frac{d}{dx}(e^x)=e^x$

32. $\lim\limits_{x\to0^+}(1+\sin(3x))^{\frac{1}{x}}$ → Type 1^∞

$\lim\limits_{x\to0^+}(1+\sin(3x))=1+\sin(0)=1$, $\lim\limits_{x\to0^+}\frac{1}{x}=\frac{1}{0}=\infty$

Set $y=\ln(1+\sin(3x))^{\frac{1}{x}}=\frac{1}{x}\ln(1+\sin(3x))$

$\lim\limits_{x\to0^+}\left[\frac{1}{x}\ln(1+\sin(3x))\right]$ → $0\cdot\infty$

$=\lim\limits_{x\to0^+}\frac{\ln(1+\sin(3x))}{x}$ → $\frac{0}{0}$

$=\lim\limits_{x\to0^+}\frac{3\cos(3x)}{1+\sin(3x)}=\frac{3\cos(0)}{1+\sin(0)}=\frac{3}{1}=3$

So, $\lim\limits_{x\to\infty}(1+\sin(3x))^{\frac{1}{x}}=e^3$

$\ln[f(x)]^{g(x)}=g(x)\ln[f(x)]$

$\frac{d}{dx}(\ln[f(x)])=\frac{f'(x)}{f(x)}$

$\frac{d}{dx}(\sin(f(x)))=f'(x)\cos(f(x))$

33. $\lim\limits_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$ → Type 1^∞

$\lim\limits_{x\to\infty}\frac{2x-3}{2x+5}=\frac{2}{2}=1$, $\lim\limits_{x\to\infty}(2x+1)=\infty$

Set $y=\ln\left(\frac{2x-3}{2x+5}\right)^{2x+1}=(2x+1)\ln\left(\frac{2x-3}{2x+5}\right)$

$\lim\limits_{x\to\infty}\left[(2x+1)\ln\left(\frac{2x-3}{2x+5}\right)\right]$ → $0\cdot\infty$

$=\lim\limits_{x\to\infty}\frac{\ln\left(\frac{2x-3}{2x+5}\right)}{\frac{1}{2x+1}}$ → $\frac{0}{0}$

$=\lim\limits_{x\to\infty}\frac{\ln(2x-3)-\ln(2x+5)}{\frac{1}{2x+1}}$

$=\lim\limits_{x\to\infty}\frac{\frac{2}{2x-3}-\frac{2}{2x+5}}{-\frac{2}{(2x+1)^2}}$

$=\lim\limits_{x\to\infty}\frac{\frac{2(2x+5)-2(2x-3)}{(2x-3)(2x+5)}}{-\frac{2}{(2x+1)^2}}$

$=\lim\limits_{x\to\infty}\frac{-8(2x+1)^2}{(2x-3)(2x+5)}=\lim\limits_{x\to\infty}\frac{-32x^2-32x-8}{4x^2+4x-15}=-\frac{32}{4}=-8$

So, $\lim\limits_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}=e^{-8}$

$\ln[f(x)]^{g(x)}=g(x)\ln[f(x)]$

$\ln\left(\frac{f(x)}{g(x)}\right)=\ln[f(x)]-\ln[g(x)]$

$\frac{d}{dx}(\ln[f(x)])=\frac{f'(x)}{f(x)}$

$\frac{d}{dx}(\sin(f(x)))=f'(x)\cos(f(x))$

If we have $f(x)=\frac{ax^m+\cdots}{bx^n+\cdots}$:

1. If $m>n$, then $\lim\limits_{x\to\infty}f(x)=\pm\infty$

2. If $m=n$, then $\lim\limits_{x\to\infty}f(x)=\frac{a}{b}$

3. If $m<n$, then $\lim\limits_{x\to\infty}f(x)=0$

34. $\lim_{x\to\infty}(x - \ln x)$ → Type $\infty - \infty$

$\lim_{x\to\infty} x = \infty$, $\lim_{x\to\infty} \ln x = \infty$

$= \lim_{x\to\infty}\left[x\left(1 - \frac{\ln x}{x}\right)\right] = (\infty)(1-0) = \infty$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\lim_{x\to\infty}\frac{\ln x}{x} \to \frac{\infty}{\infty}$

$= \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \frac{1}{\infty} = 0$

35. $\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}$ → Type $\frac{\infty}{\infty}$

$= \lim_{x\to-\infty}\frac{1}{\frac{2x}{2\sqrt{x^2+1}}}$

$\frac{d}{dx}\left(\sqrt{f(x)}\right) = \frac{f'(x)}{2\sqrt{f(x)}}$

$= \lim_{x\to-\infty} -\frac{\sqrt{x^2+1}}{x}$

$= \lim_{x\to-\infty} -\frac{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x}$

$= \lim_{x\to-\infty} -\frac{x\sqrt{\left(1+\frac{1}{x^2}\right)}}{x} = \lim_{x\to-\infty} -\frac{\sqrt{\left(1+\frac{1}{x^2}\right)}}{1} = -\sqrt{1+\frac{1}{\infty}} = -1$

❖ **If** $\lim_{x\to0}\frac{\sin(ax)+bx^3-2x}{x^3} = 0$. **Find a and b.** → Type $\frac{0}{0}$

$\lim_{x\to0}\frac{a\cos(ax)+3bx^2-2}{3x^2} \to \frac{0}{0}$

$\to a\cos(0) + 3b(0) - 2 = 0 \ \to a - 2 = 0 \ \to a = 2$

$= \lim_{x\to0}\frac{2\cos(2x)+3bx^2-2}{3x^2}$

$= \lim_{x\to0}\frac{-4\sin(2x)+6bx}{6x} \to \frac{0}{0}$

$= \lim_{x\to0}\frac{-8\cos(2x)+6b}{6} = \frac{-8+6b}{6} = 0$

$\to -8 + 6b = 0 \ \to 6b = 8 \ \to b = \frac{8}{6} = \frac{4}{3}$