OL Academy

Lesson 3

MATHS101

1.5 Inverse functions and logarithms

2.2 The Limit of a Function

2.3 Calculating Limits using the Limit Laws

1.5 Inverse Functions and Logarithms

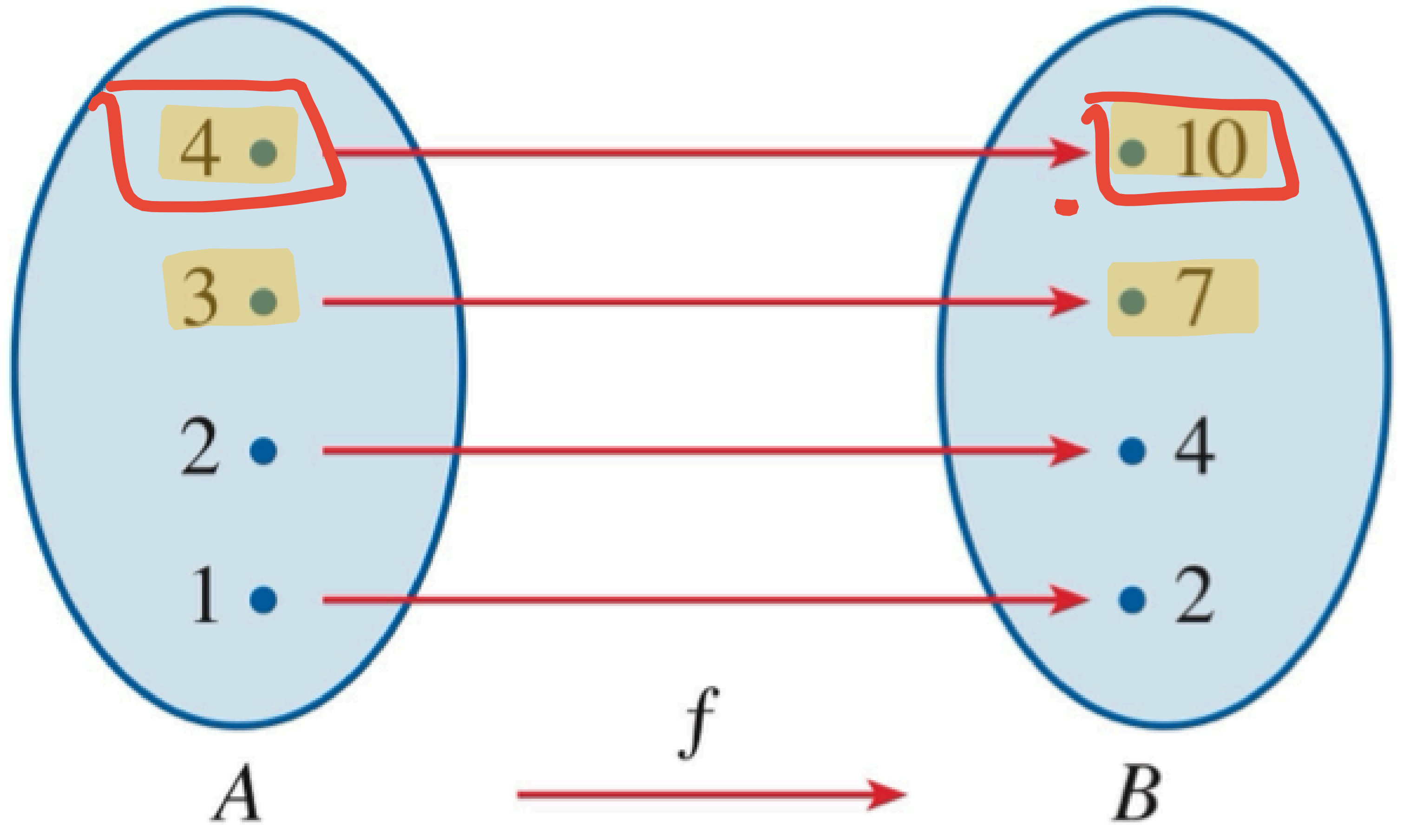
1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

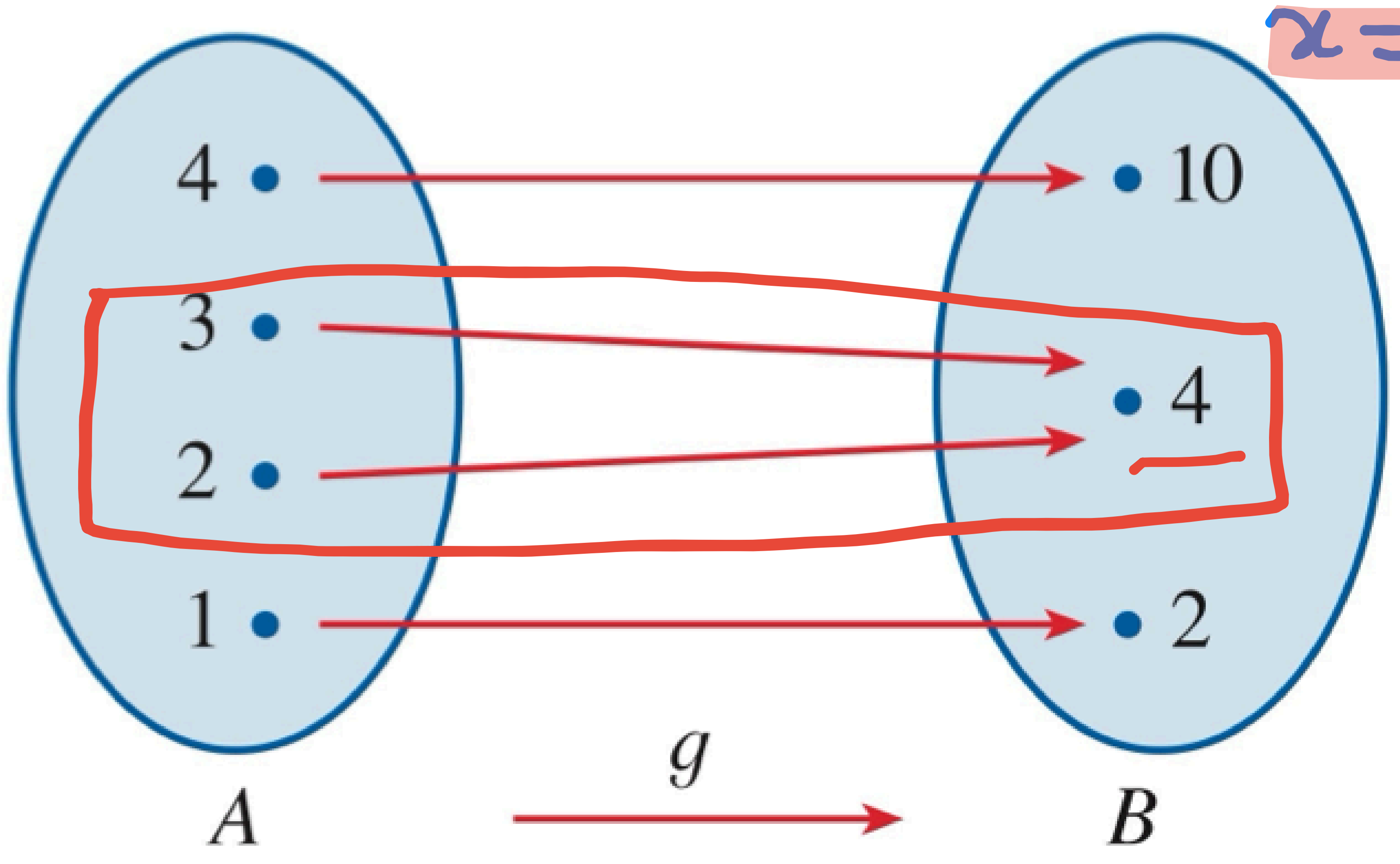
$x^2+2 \Rightarrow$ $x = \boxed{2} \rightarrow (2)^2+2 = \underline{6}$

$x = \boxed{-2} \rightarrow (-2)^2+2 = \underline{6}$

x $\qquad$ $f(x)$

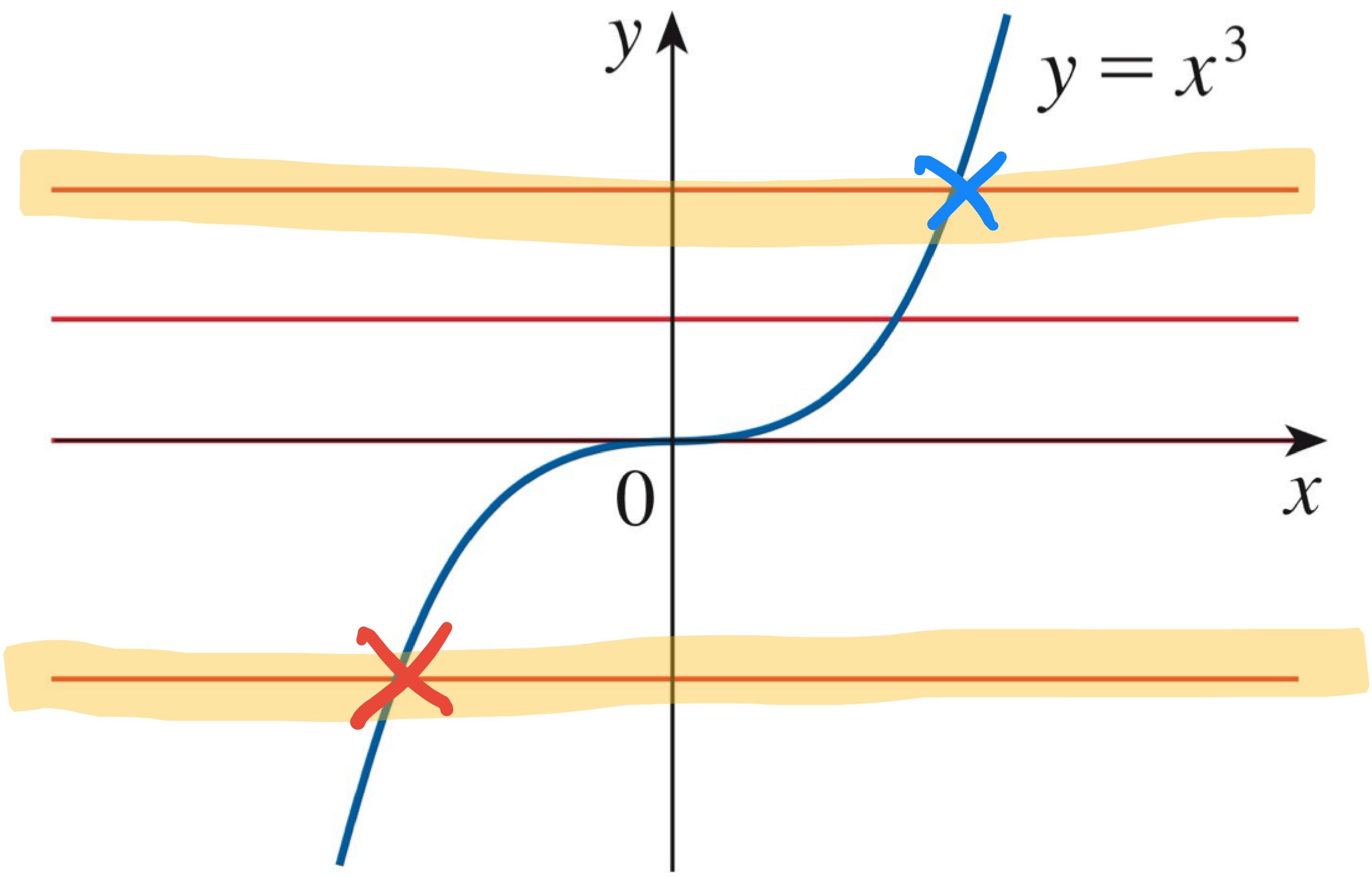


Domain $\qquad$ Range



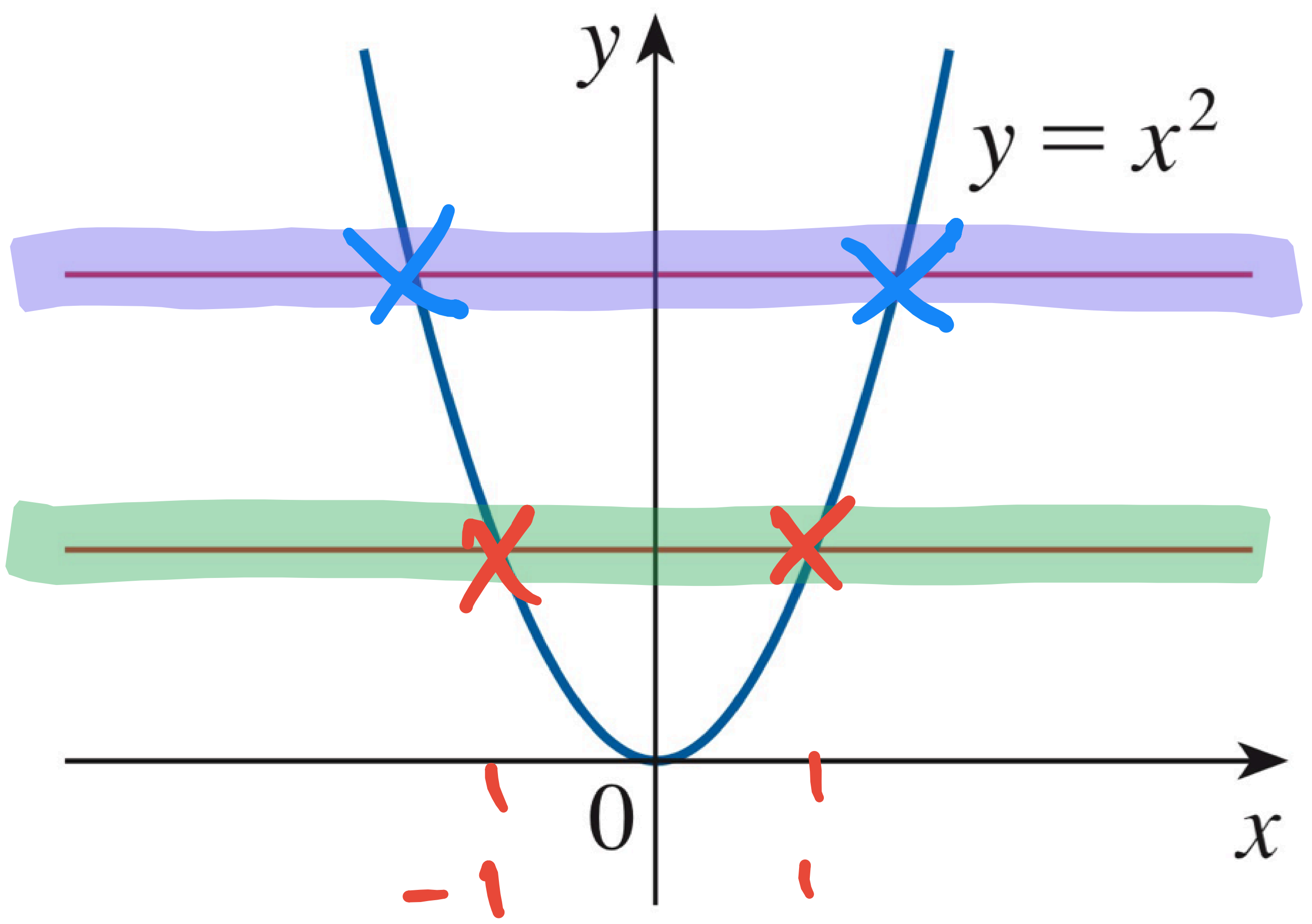
Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

EXAMPLE 1 Is the function $f(x) = x^3$ one-to-one?



one to one

EXAMPLE 2 Is the function $g(x) = x^2$ one-to-one?



$(-1)^2 = 1$ $(1)^2 = 1$

$\boxed{5}$ How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Find the inverse of the following:

a) $f(x) = 3x + 5$

① $y = 3x + 5$

② $x = 3y + 5$

$(3y = x - 5) \div 3 \Rightarrow y = \dfrac{x-5}{3}$

$f^{-1}(x) = \dfrac{x-5}{3}$

b) $f(x) = \dfrac{3x-7}{4x+3}$

① $y = \dfrac{3x-7}{4x+3}$

② $x = \dfrac{3y-7}{4y+3} \rightarrow x(4y+3) = 3y-7$

$x(4y+3) = 3y-7 \Rightarrow 4xy + 3x = 3y - 7$

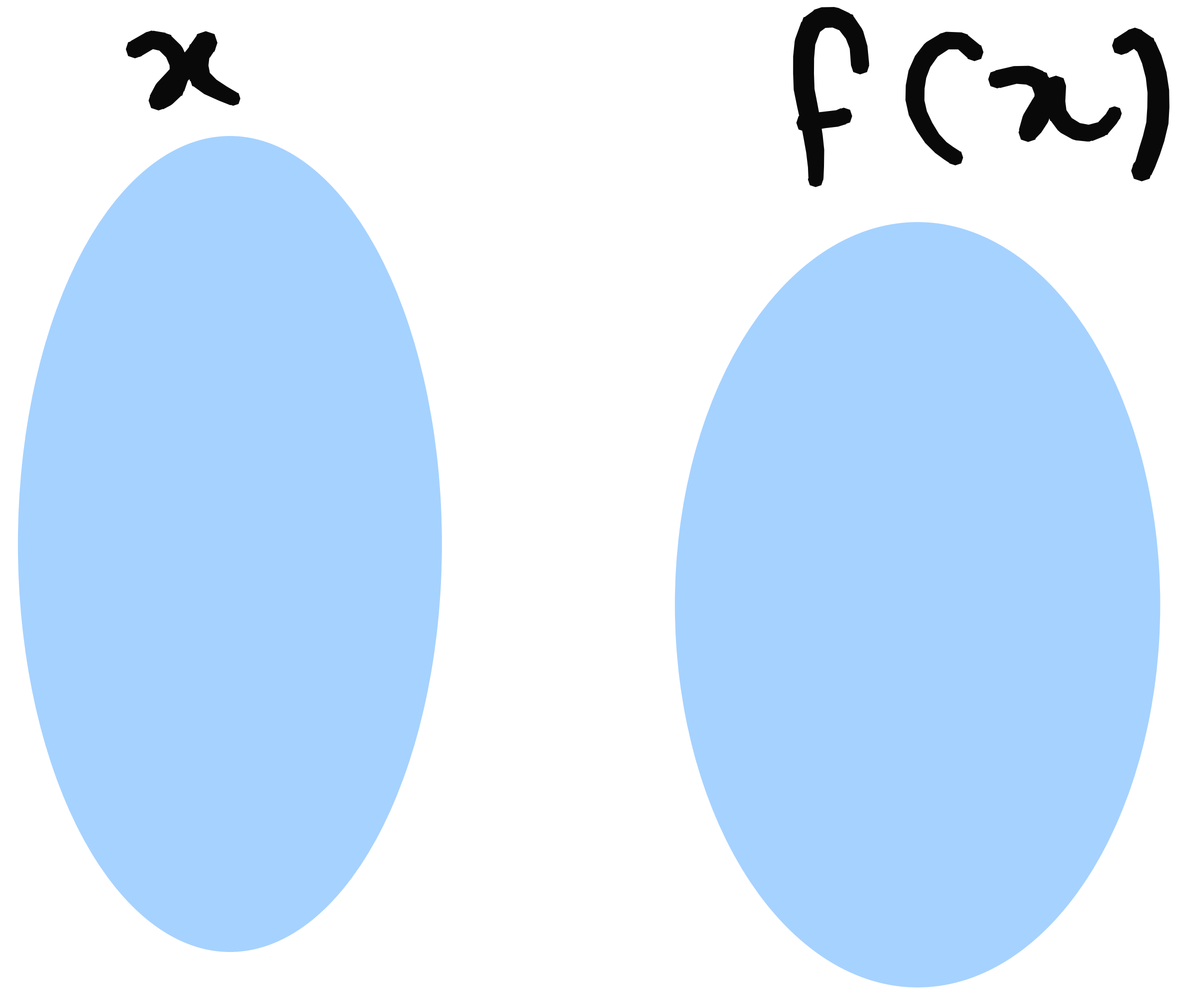
$4xy - 3y = -3x - 7 \Rightarrow \left(y(4x-3) = -3x-7\right) \div 4x-3$

$y = \dfrac{-3x-7}{4x-3} \Rightarrow f^{-1}(x) = \dfrac{-3x-7}{4x-3}$

c) $f(x) = \sqrt{2x-6}$

domain of f^{-1} = range of f

range of f^{-1} = domain of f



Domain $\leftrightarrow$ Range

$[3, \infty)$, $[0, \infty)$

$2x - 6 \geq 0 \rightarrow 2x \geq 6 \rightarrow x \geq 3$

$y = \sqrt{2x-6} \rightarrow x = \sqrt{2y-6}$

$\left(x = (2y-6)^{\frac{1}{2}}\right)^2 \Rightarrow x^2 = 2y - 6$

$2y = x^2 + 6 \Longrightarrow y = \frac{x^2+6}{2} \Longrightarrow y = \frac{x^2}{2} + 3$

$f^{-1}(x) = \frac{1}{2}x^2 + 3$

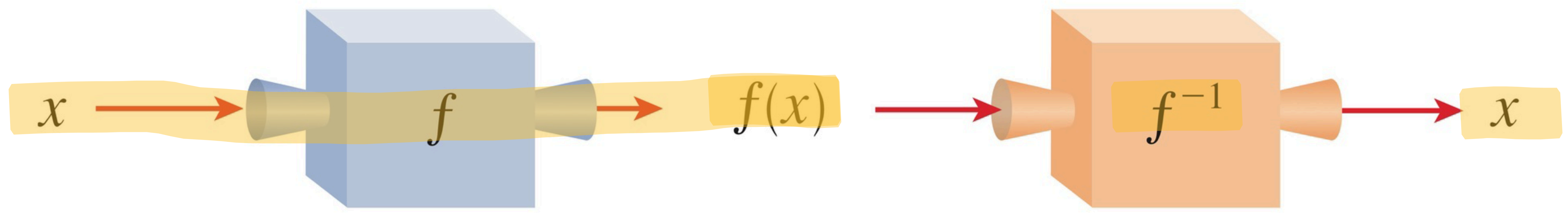
$f(x) = \sqrt{2x-6}$

$f(4) = \sqrt{2(4)-6}$

$f(4) = \sqrt{2}$

$f^{-1}(\sqrt{2}) = \frac{(\sqrt{2})^2}{2} + 3$

$f^{-1}(\sqrt{2}) = 1 + 3 = 4$



$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

d) $f(x) = \sqrt[3]{2x-7}$ ① $y = \sqrt[3]{2x-7}$ ② $\left(x = \sqrt[3]{2y-7}\right)^3$

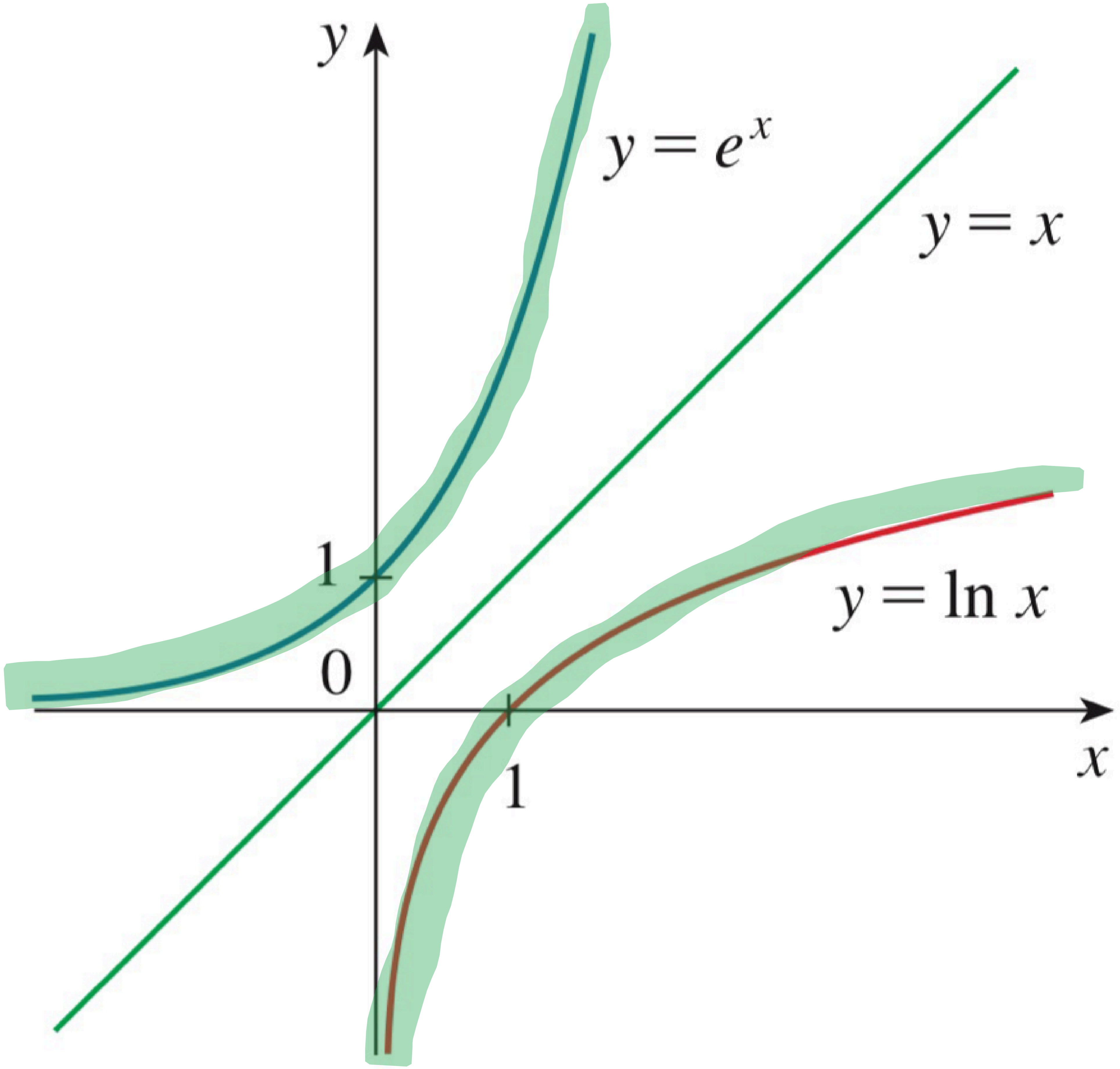
$$x^3 = 2y-7 \Rightarrow 2y = x^3+7 \Rightarrow y = \frac{x^3+7}{2} \Rightarrow f^{-1}(x) = \frac{x^3+7}{2}$$

$$f^{-1}(f(x)) = \frac{\left(\cancel{\sqrt[3]{2x-7}}\right)^{\cancel{3}}+7}{2} = \frac{(2x-\cancel{7})+\cancel{7}}{2}$$

$$\frac{\cancel{2}x}{\cancel{2}} = x$$

How does the inverse look like?

$\log_e x = \ln x$



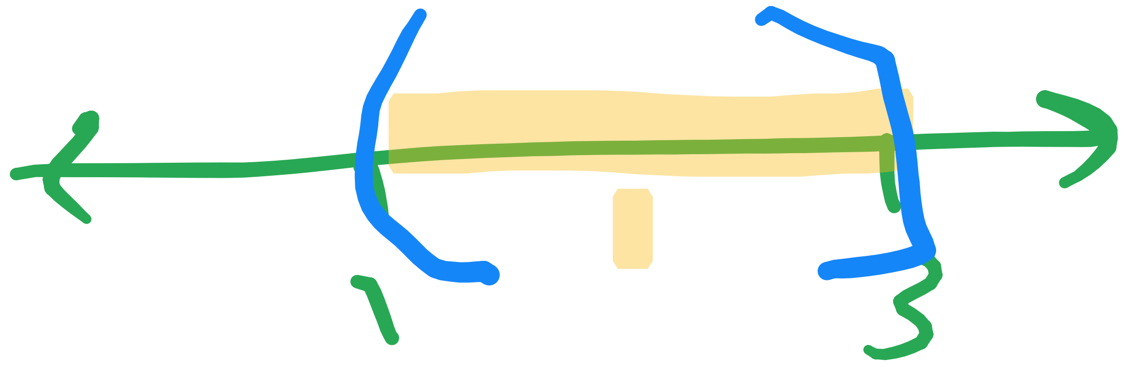
$y = e^x \Rightarrow$

Domain $(-\infty, \infty)$, Range $(0, \infty)$

$y = \ln x \Rightarrow$

Domain $(0, \infty)$, Range $(-\infty, \infty)$

$(1, 3]$ 1.0001

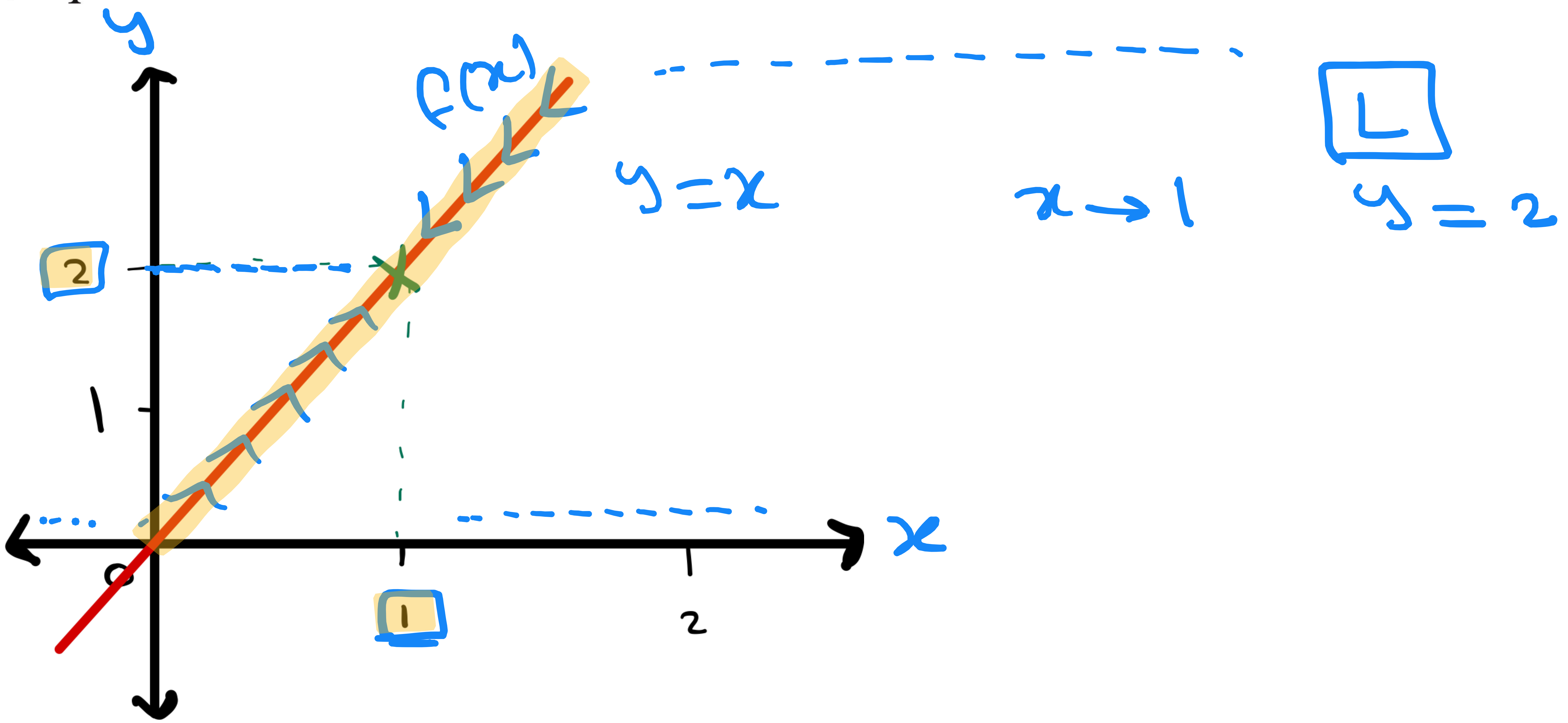


2.2 | The Limit of a Function

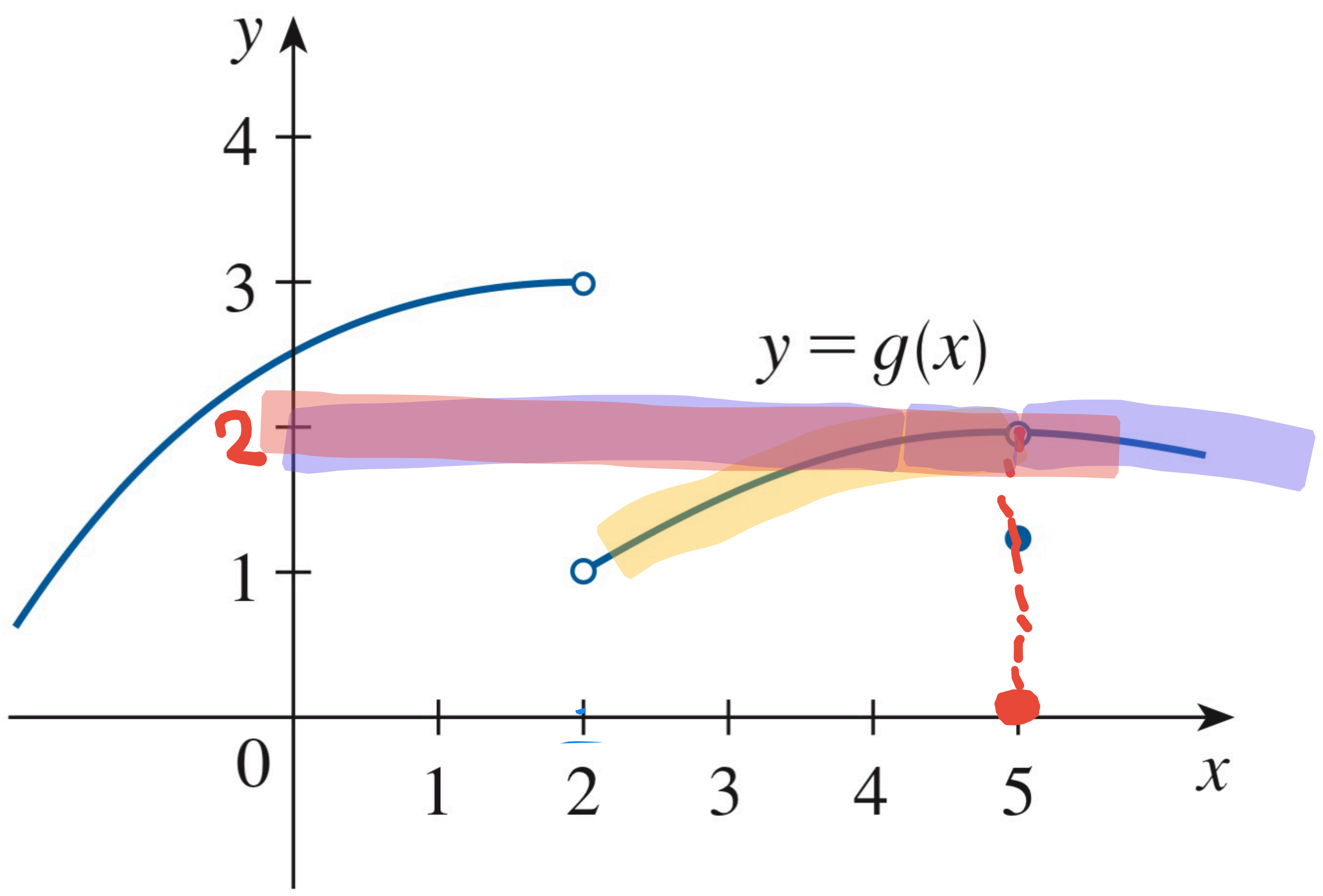
$$\lim_{x \to a} f(x) = L$$

"the limit of $f(x)$, as x approaches a, equals L"

x	$f(x)$
0.9	1.9
0.99	1.99
0.999	1.999
1.001	2.001
1.01	2.01
1.1	2.1



3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$



Use the graph to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ $=3$

(b) $\lim_{x \to 2^+} g(x)$ $=1$

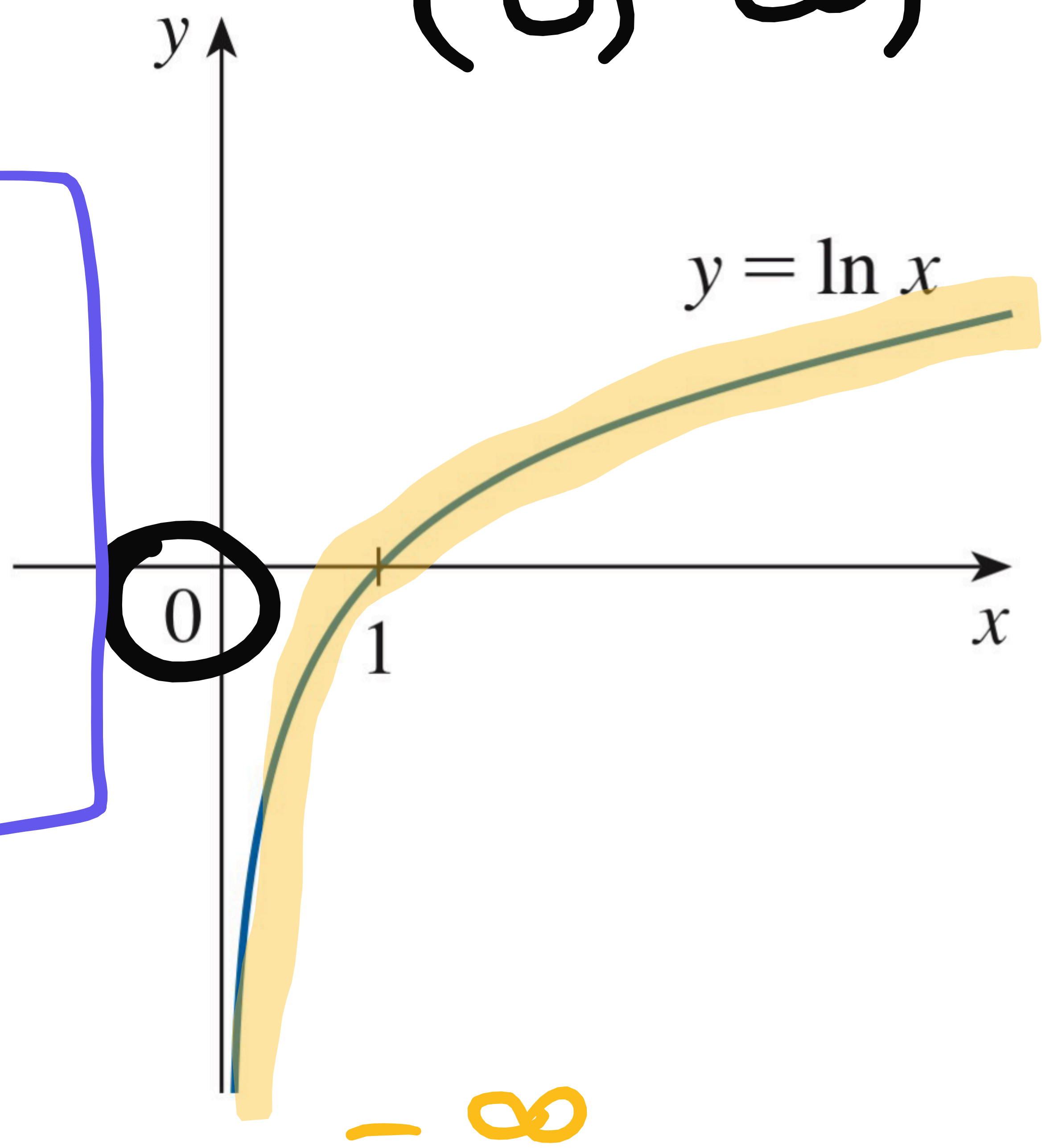
(c) $\lim_{x \to 2} g(x)$ $=$ DNE

(d) $\lim_{x \to 5^-} g(x)$ $=2$

(e) $\lim_{x \to 5^+} g(x)$ $=2$

(f) $\lim_{x \to 5} g(x)$ $=2$

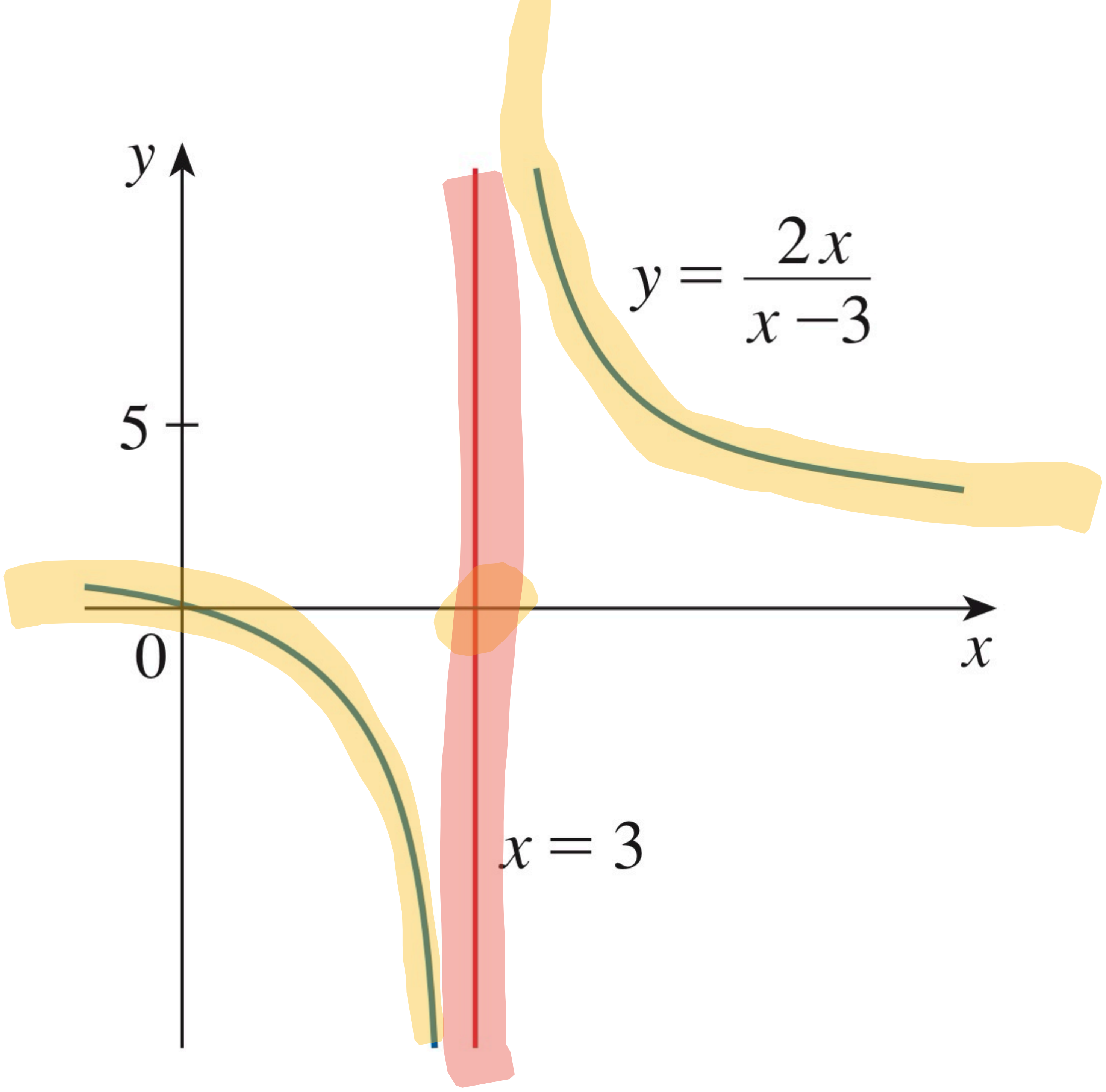
$(0, \infty)$



$-\infty$

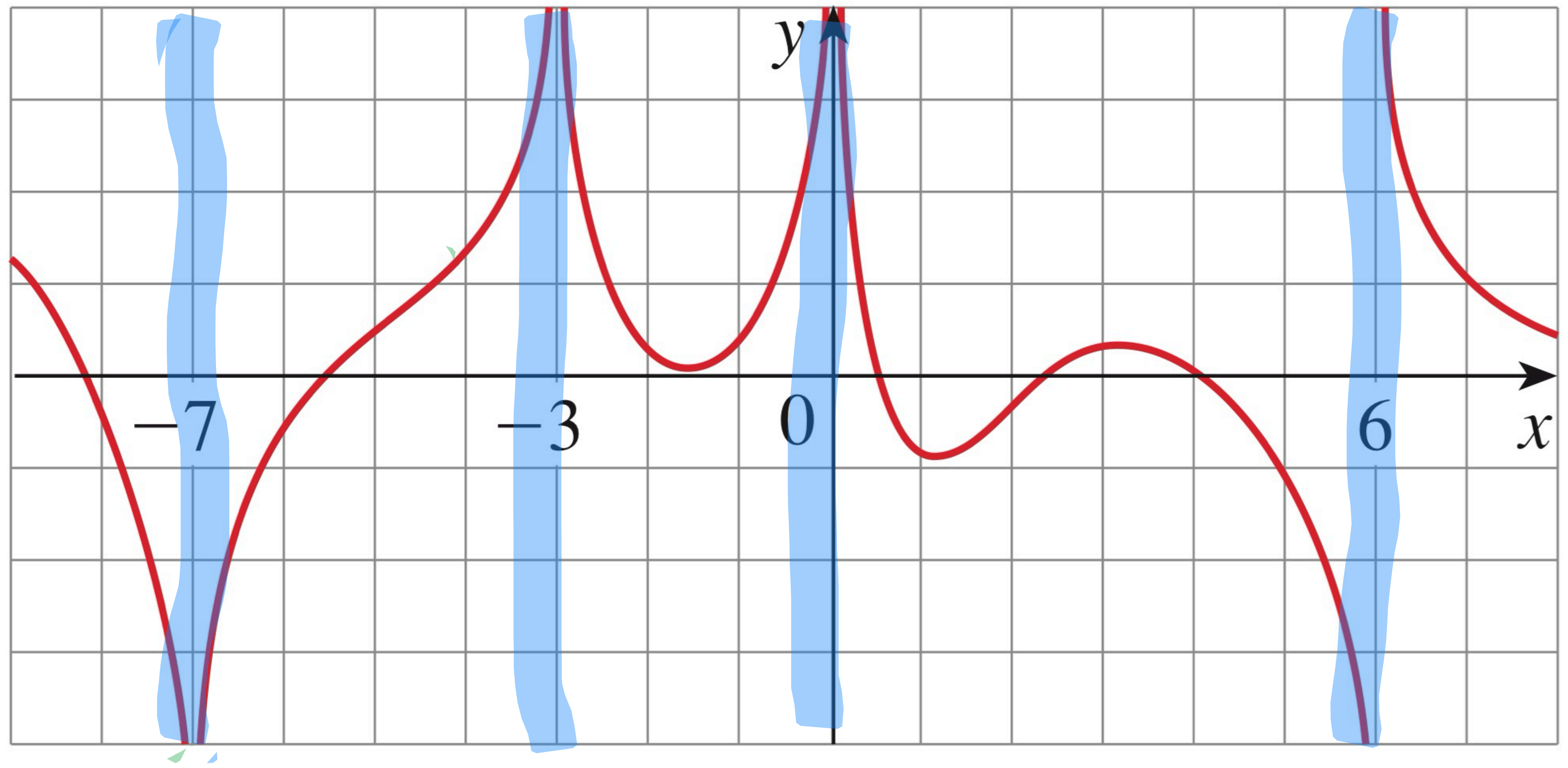
$$\lim_{x \to 0^+} \ln x = \boxed{-\infty}$$

$$\lim_{x \to 0} \ln x = DNE$$



$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty \qquad \lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$$

the line $x = 3$ is a vertical asymptote.



$f(x)$

9. For the function f whose graph is shown, state the following.

(a) $\lim\limits_{\boxed{x \to -7}} f(x)$

$= -\infty$

(b) $\lim\limits_{x \to -3} f(x)$

$= \infty$

(c) $\lim\limits_{x \to 0} f(x)$

$= \infty$

(d) $\lim\limits_{x \to 6^-} f(x)$

$= -\infty$

(e) $\lim\limits_{x \to 6^+} f(x)$

$= \infty$

$\lim\limits_{x \to 6} f(x) = \text{DNE}$

(f) The equations of the vertical asymptotes

$x = -7$, $x = -3$, $x = 0$ and $x = 6$

2.3 Calculating Limits Using the Limit Laws

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$

4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

1 2 3 0.5

0.7

6. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer

7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x\to a} f(x) > 0$.]

$$\lim_{x\to 5} 10 = 10$$

8. $\lim_{x\to a} c = c$

9. $\lim_{x\to a} x = a$

10. $\lim_{x\to a} x^n = a^n$ where n is a positive integer

$$\lim_{x\to 5} x^2 = 5^2 = 25$$

11. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

EXAMPLE 1 Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$

$\lim_{x \to -2} f(x) = 1$, $\lim_{x \to -2} g(x) = -1$

$\lim_{x \to -2} f(x) + \lim_{x \to -2} [5\, g(x)]$

$\boxed{\lim_{x \to -2} f(x)} + 5 \boxed{\lim_{x \to -2} g(x)}$

$1 + 5(-1) = 1 - 5 = -4$

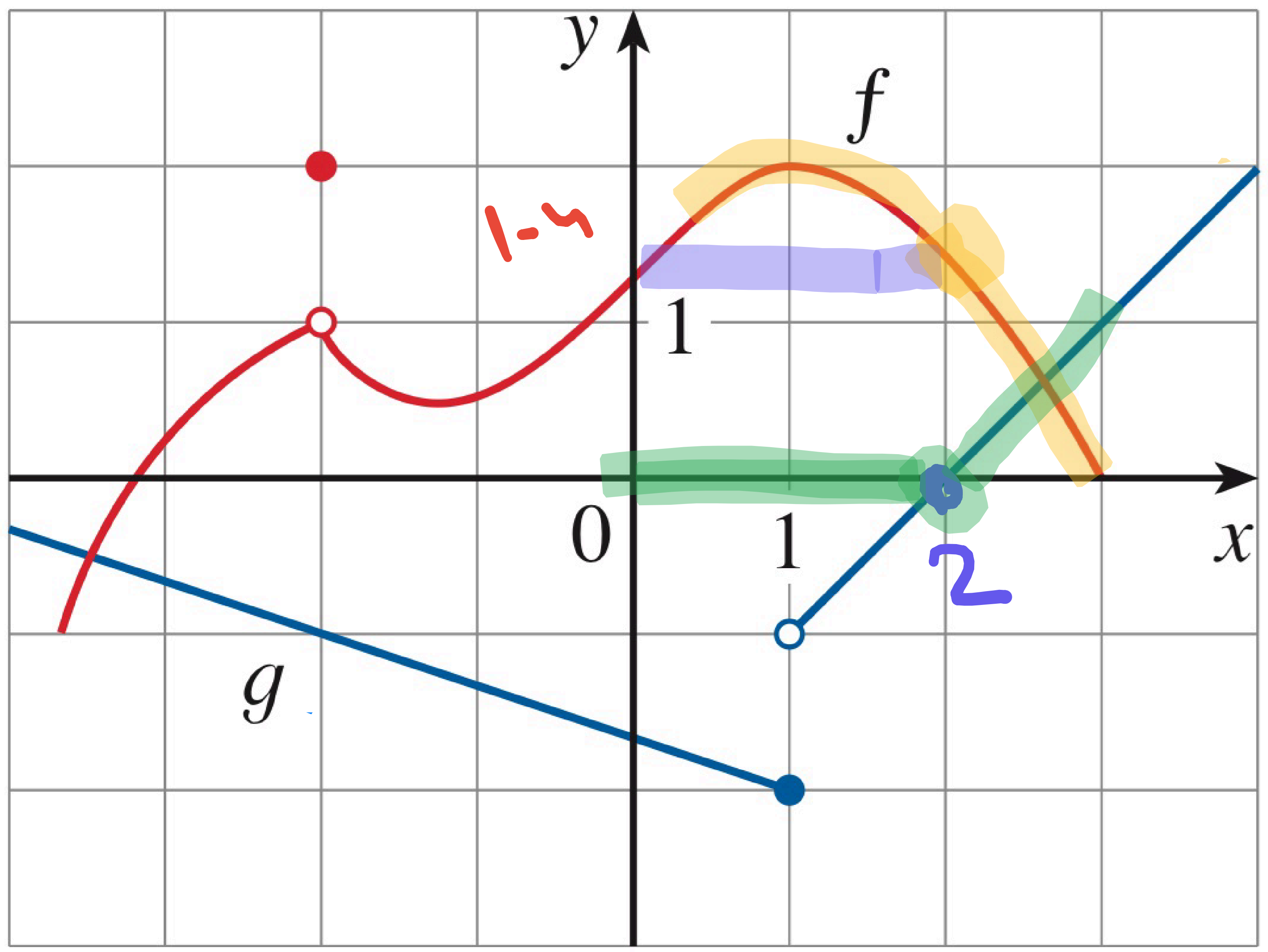


FIGURE 1

(c) $\lim_{x \to 2} \dfrac{f(x)}{g(x)}$

$\lim_{x \to 2} f(x) \approx 1.4$, $\lim_{x \to 2} g(x) = 0$

$\lim_{x \to 2} \dfrac{f(x)}{g(x)} = \text{DNE} \rightarrow \lim_{x \to 2} g(x) = 0$

(b) $\lim_{x \to 1} [f(x)g(x)]$

$\lim_{x \to 1} f(x) = 2$

$\lim_{x \to 1} g(x) = \text{DNE}$

$\lim_{x \to 1^+} g(x) = -1$ $\lim_{x \to 1^-} g(x) = -2$

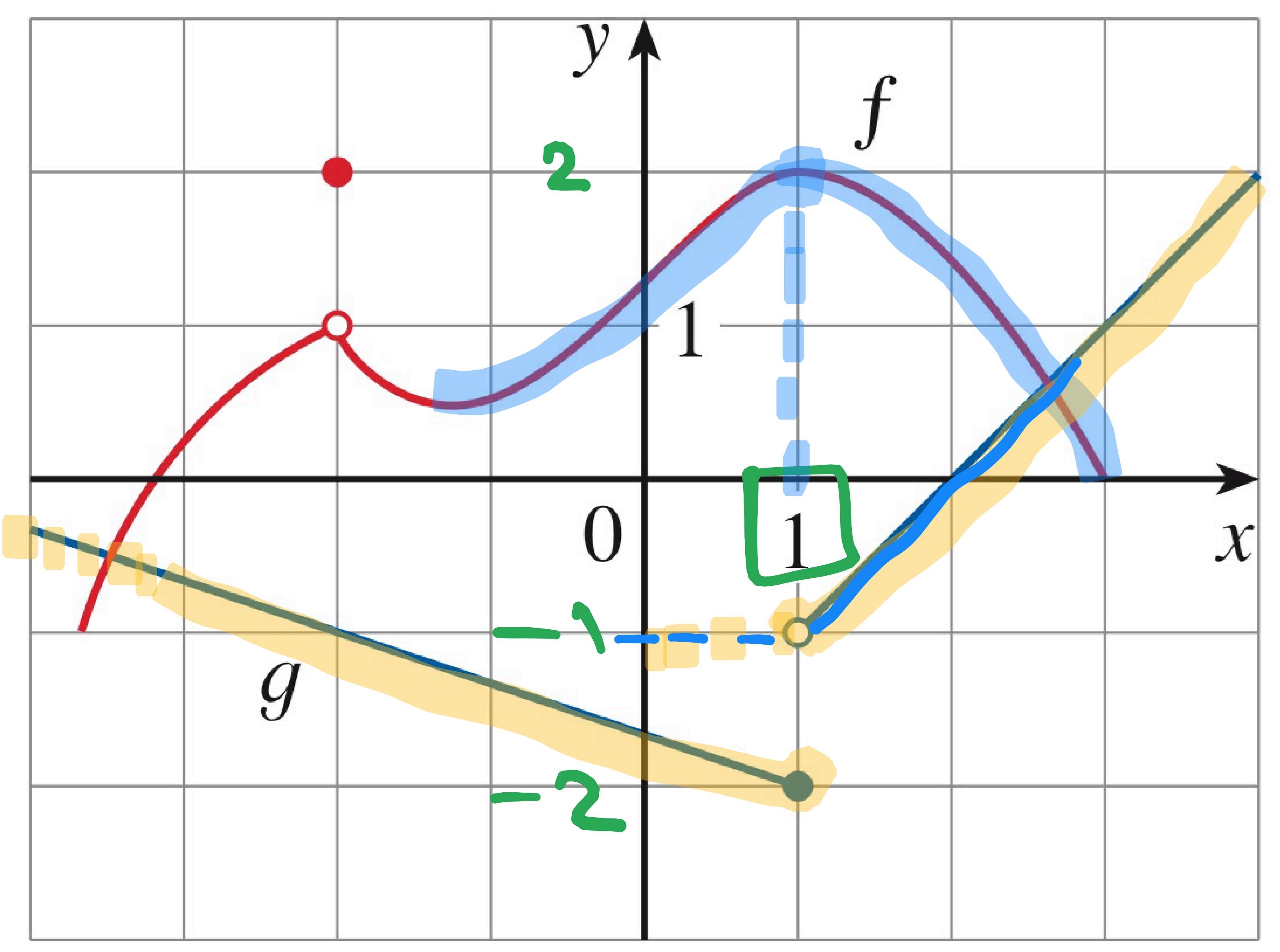


FIGURE 1

$\lim_{x \to 1^+} [f(x)\,g(x)] = \lim_{x \to 1^+} f(x) \cdot g(x) = (2) \cdot (-1) = -2$

$\lim_{x \to 1^-} [f(x)\,g(x)] = \lim_{x \to 1^-} f(x) \cdot g(x) = (2)\,(-2) = -4$

Evaluate the limit, if it exists.

11. $\lim_{x \to -2} (3x - 7)$

13. $\lim_{t \to 4} \dfrac{t^2 - 2t - 8}{t - 4}$

19. $\displaystyle\lim_{t \to 3} \frac{t^3 - 27}{t^2 - 9}$

22. $\displaystyle\lim_{x \to 9} \frac{9 - x}{3 - \sqrt{x}}$

28. $\displaystyle\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$

33. $\displaystyle\lim_{h \to 0} \frac{(x + h)^3 - x^3}{h}$

The Squeeze Theorem

3 **The Squeeze Theorem** If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

39. If $4x - 9 \leqslant f(x) \leqslant x^2 - 4x + 7$ for $x \geqslant 0$, find $\lim_{x \to 4} f(x)$.