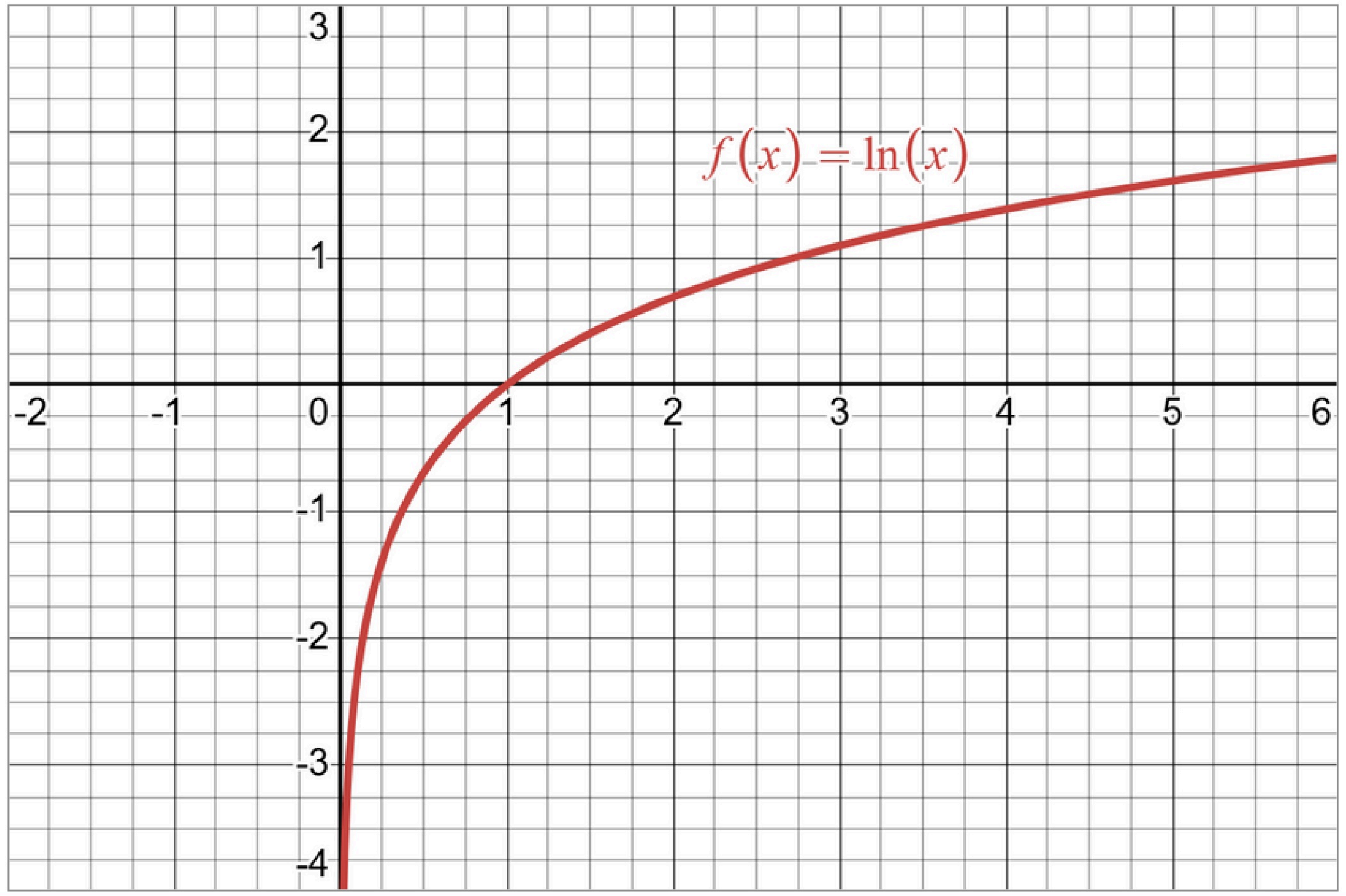
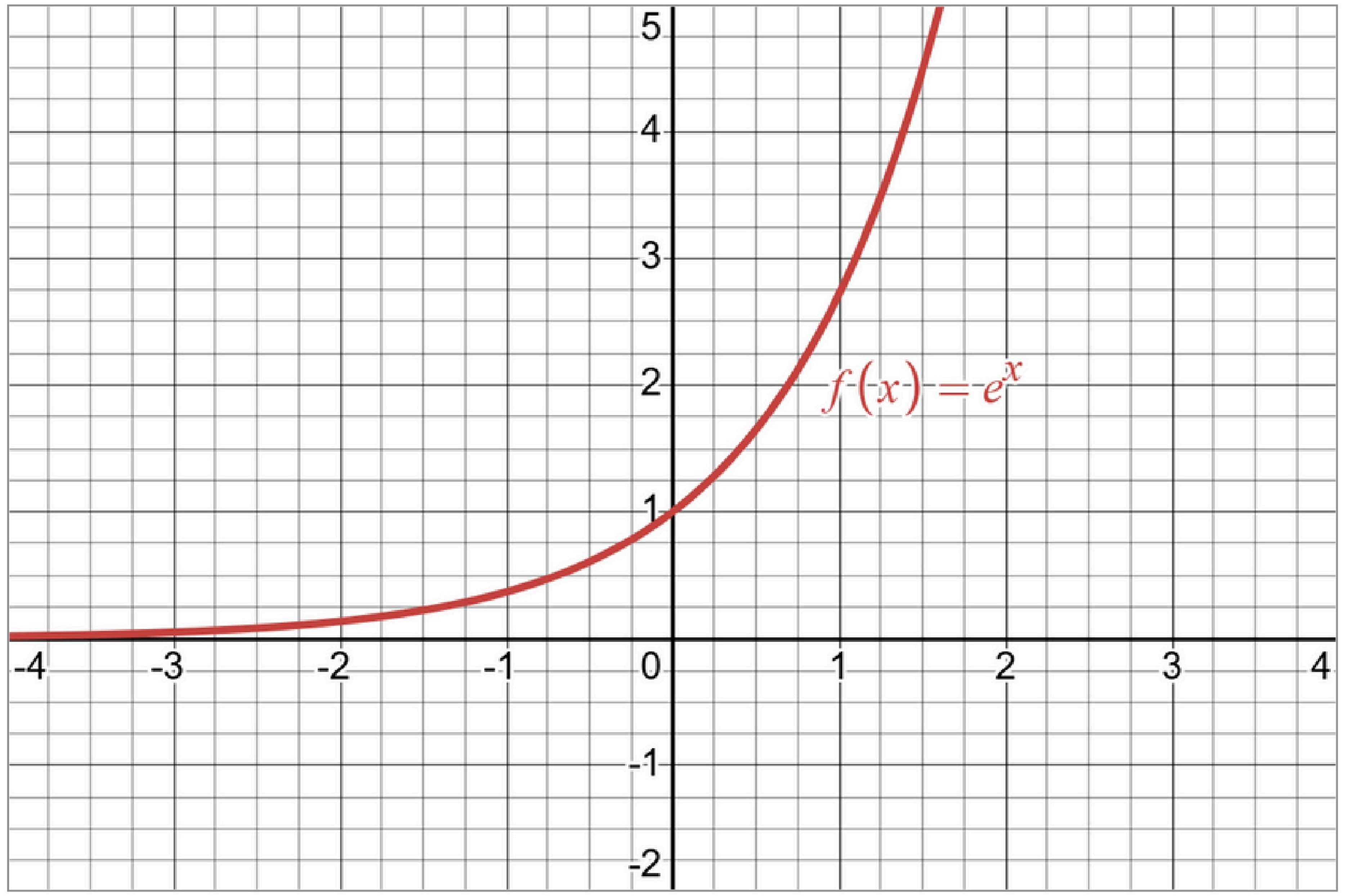
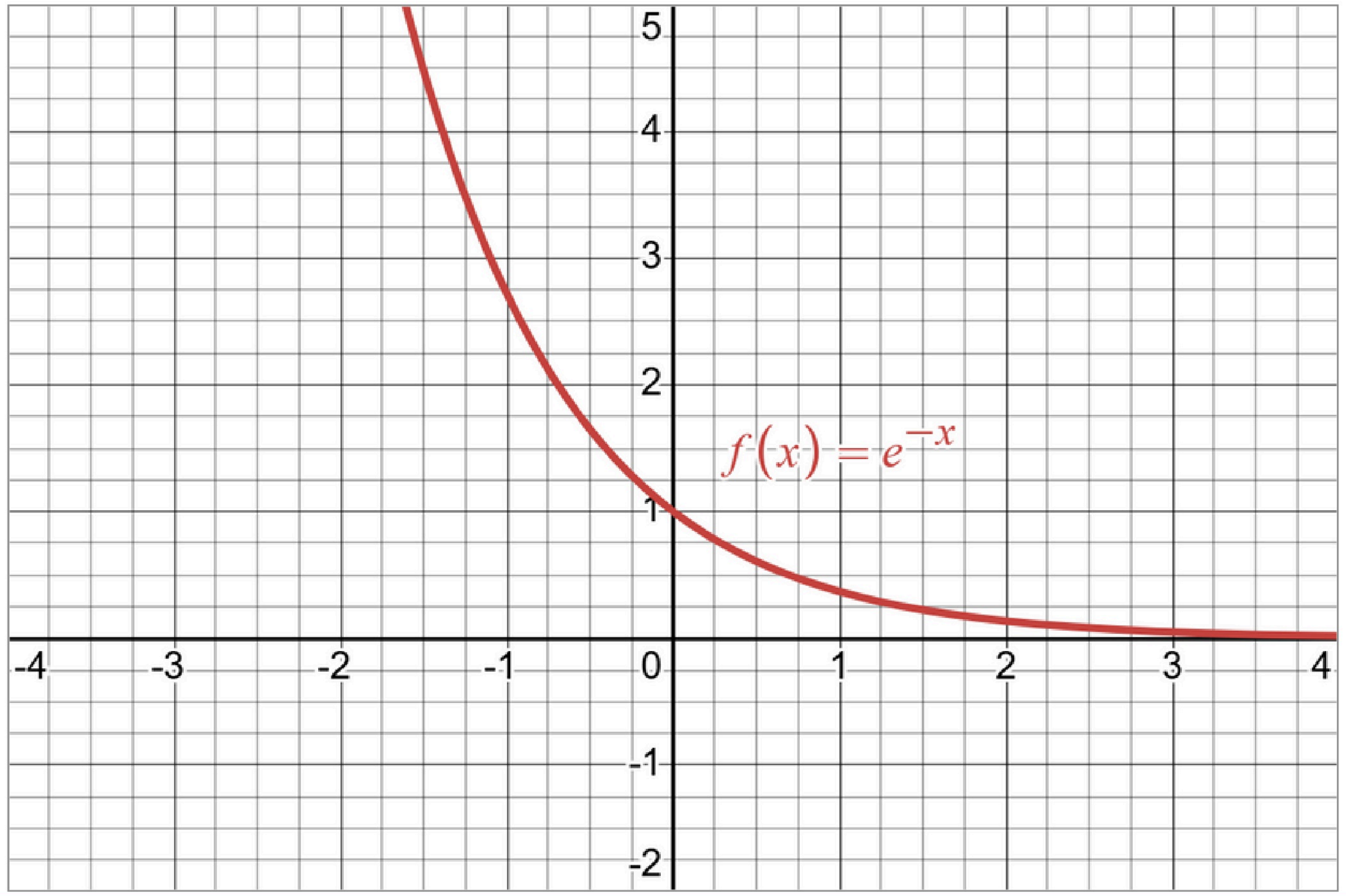
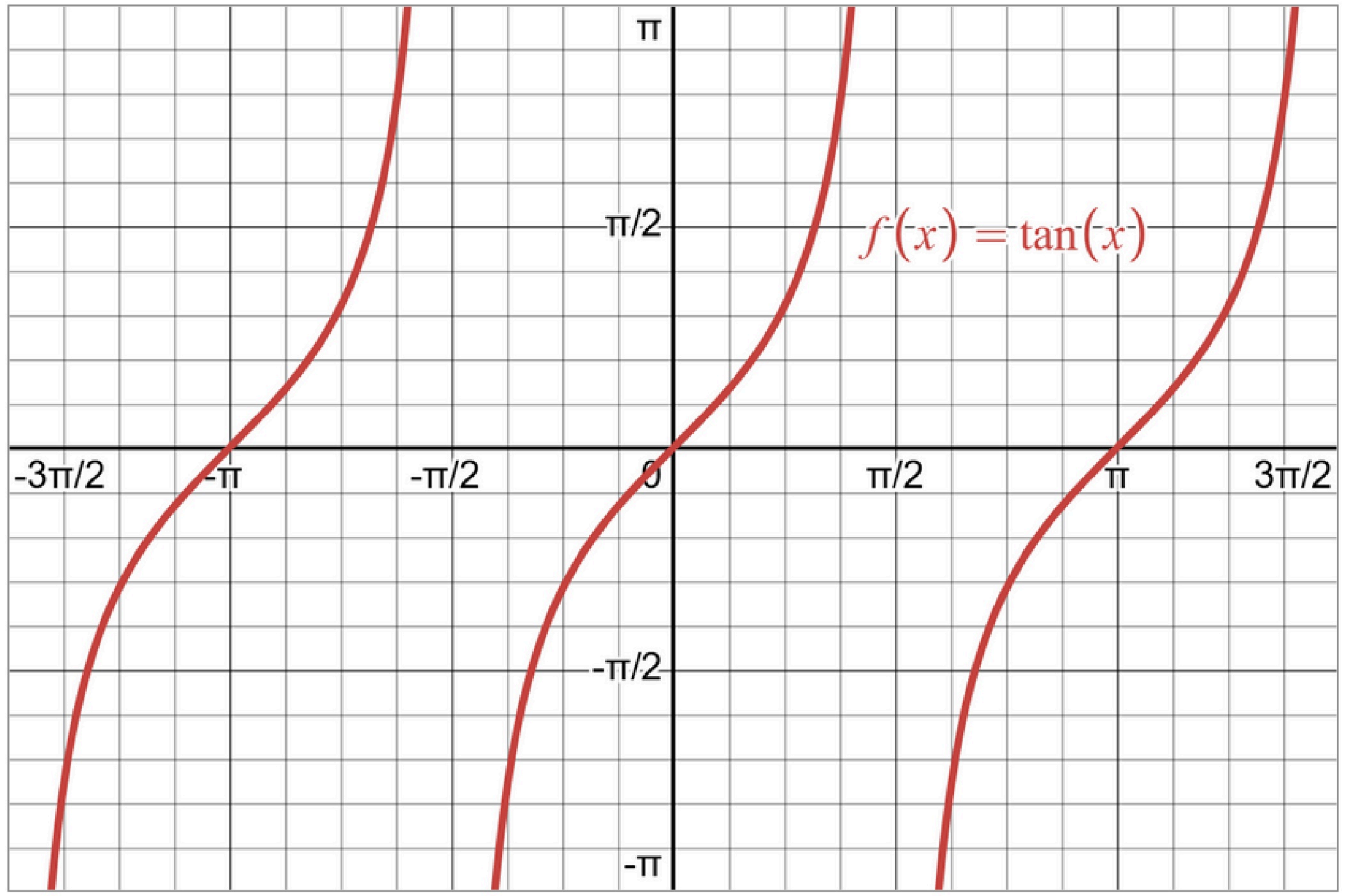
OL Academy

MATHS102

Lesson 1

4.4 L'Hospital's Rule

Function	Domain	Range	End Behavior
	$(0, \infty)$	$(-\infty, \infty)$	$\lim_{x \to \infty} \ln(x) = \infty$ $\lim_{x \to 0^+} \ln(x) = -\infty$
	$(-\infty, \infty)$	$(0, \infty)$	$\lim_{x \to \infty} e^x = \infty$ $\lim_{x \to -\infty} e^x = 0$

<table>
<tr><th>Function</th><th>Domain</th><th>Range</th><th>End Behavior</th></tr>
<tr><td>
</td><td>$(-\infty, \infty)$</td><td>$(0, \infty)$</td><td>$\lim_{x \to \infty} e^{-x} = 0$
$\lim_{x \to -\infty} e^{-x} = \infty$</td></tr>
<tr><td>
</td><td>$x \neq \frac{\pi}{2} + n\pi$</td><td>$(-\infty, \infty)$</td><td>$\lim_{x \to \pm\frac{\pi}{2}^{+}} \tan x = -\infty$
$\lim_{x \to \pm\frac{\pi}{2}^{-}} \tan x = \infty$</td></tr>
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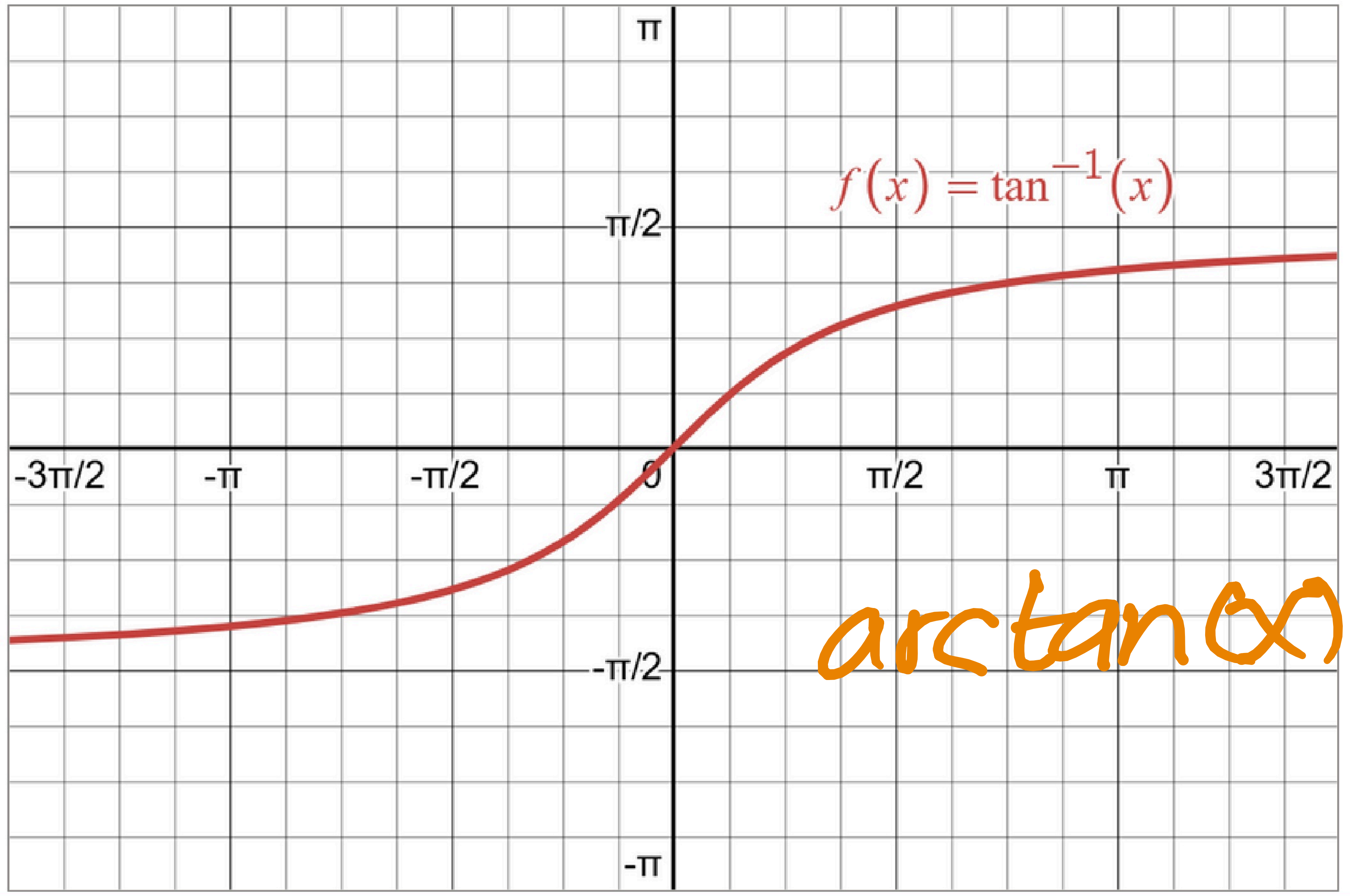
Function	Domain	Range	End Behavior
	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$ $\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$
 **Exponential Functions** $b^x, b > 1$	$(-\infty, \infty)$	$(0, \infty)$	$\lim_{x\to\infty} b^x = \infty$ $\lim_{x\to-\infty} b^x = 0$
 **Exponential Functions** $b^x, 0 < b < 1$	$(-\infty, \infty)$	$(0, \infty)$	$\lim_{x\to\infty} b^x = 0$ $\lim_{x\to-\infty} b^x = \infty$

Indeterminate Forms

Indeterminate Fractions

$$\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$$

Indeterminate Products

$$(0 \cdot \infty)$$

Indeterminate Differences

$$(\infty - \infty)$$

Indeterminate Powers

$$(0^0, \infty^0, 1^\infty)$$

$\frac{\infty}{0^+} = \frac{a}{0^+} = \infty$, $\frac{0}{\infty} = \frac{a}{\infty} = 0$, $\infty + \infty = \infty$, $0^\infty = 0$

$\frac{\infty}{0^-} = \frac{a}{0^-} = -\infty$, $\infty \cdot \infty = \infty$, $-\infty - \infty = -\infty$, $\infty^\infty = \infty$

01

Indeterminate Fractions $\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$ ----→ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

a can be any real number or ∞

✔ **Apply L'Hospital's Rule:**

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

⚠ we can apply L'Hospital's Rule more than once.

Examples

1. $$\lim_{x \to 0} \frac{\sin x}{x} \rightarrow \frac{\sin(0)}{0} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\cos x}{1}$$

$$= \frac{\cos(0)}{1} = \frac{1}{1} = 1$$

(2) $\lim_{x\to\infty} \frac{e^x}{x^2} \to \frac{\infty}{\infty^2} = \frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{e^x}{2x} \to \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{e^x}{2}$$

$$= \frac{\infty}{2} = \infty$$

(3) $\lim_{x\to 0} \frac{x \cdot 3^x}{3^x - 1} \to \frac{0 \cdot 3^0}{3^0 - 1} = \frac{0}{0}$

$$= \lim_{x\to 0} \frac{(1)\cdot 3^x + x \cdot 3^x \cdot \ln(3)}{3^x \ln(3)}$$

$$= \lim_{x\to 0} \frac{\cancel{3^x}(1 + x\ln(3))}{\cancel{3^x}\ln(3)}$$

$$= \lim_{x\to 0} \frac{1 + x\ln(3)}{\ln(3)}$$

$$= \frac{1}{\ln(3)}$$

02

Indeterminate Products $(0 \cdot \infty)$

$$\lim_{x\to a} \overset{0}{f(x)} \cdot \overset{\infty}{g(x)} \dashrightarrow \text{is of the form } 0\cdot\infty$$

a can be any real number or ∞

- **Rewrite the Product as a Quotient**

$$f(x)\cdot g(x) \to \frac{f(x)}{\frac{1}{g(x)}} \quad or \quad \frac{g(x)}{\frac{1}{f(x)}}$$

- **Apply L'Hospital's Rule**

Examples

4. $$\lim_{x\to \frac{\pi}{2}^-} \tan x \ln(\sin x) \to 0\cdot\infty$$

$$= \lim_{x\to \frac{\pi}{2}^-} \frac{\ln(\sin x)}{\cot x} \to \frac{0}{0}$$

$$= \lim_{x\to \frac{\pi}{2}^-} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x}$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{\cos x}{\cancel{\sin x}} \cdot -\sin^{\cancel{2}\,1} x$$

$$= \lim_{x\to \frac{\pi}{2}} -\cos x \cdot \sin x = 0$$

03

Indeterminate Differences $(\infty - \infty)$

$$\lim_{x\to a} f(x) - g(x)$$ is of the form $\infty - \infty$

a can be any real number or ∞

- Rewrite the Difference as a Quotient or a Product

$$f(x) - g(x) \to \frac{A}{B} \text{ or } A \cdot B$$

- You will get one of the previous forms

Examples

5. $$\lim_{x\to 0^+}\left(\frac{1}{e^x - 1} - \frac{1}{x}\right) = \infty - \infty$$

$$= \lim_{x\to 0^+} \frac{x - (e^x - 1)}{(e^x - 1)\cdot x} \to \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{1 - e^x}{e^x \cdot x + (e^x - 1)} \to \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{-e^x}{e^x \cdot x + e^x + e^x}$$

$$= \frac{-1}{0 + 1 + 1} = -\frac{1}{2}$$

04

x^y Indeterminate Powers $(0^0, \infty^0, 1^\infty)$

$$\lim_{x\to a} [f(x)]^{g(x)}$$ is of the form 0^0 or ∞^0 or 1^∞

a can be any real number or ∞

- Transform the power to logarithm

$$[f(x)]^{g(x)} \to y = \ln [f(x)]^{g(x)}$$
$$y = g(x) \ln f(x)$$

- Evaluate the limit of y and the main limit

$$\lim_{x\to a} y = L \Rightarrow \lim_{x\to a} [f(x)]^{g(x)} = e^L$$

Examples

6. $\lim_{x\to\infty} (x)^{\frac{1}{x}} \to \infty^{\frac{1}{\infty}} = \infty^0$

set $y = \ln(x^{\frac{1}{x}}) = \frac{1}{x} \ln(x)$

$$\lim_{x\to\infty} \frac{1}{x} \ln(x) \to 0 \cdot \infty$$

$$= \lim_{x\to\infty} \frac{\ln(x)}{x} \to \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{\frac{1}{\infty}}{1} = 0$$

$$\Rightarrow \lim_{x\to\infty} (x)^{\frac{1}{x}} = e^0 = 1$$

⑦ $\lim_{x\to 1^-} (1-x)^{\ln x} \to 0^0$

set $y = \ln(x)\ln(1-x)$

$$\lim_{x\to 1^-} \ln(x)\cdot\ln(1-x) \to 0\cdot\infty$$

$$= \lim_{x\to 1^-} \frac{\ln(1-x)}{1/\ln(x)} \to \frac{\infty}{\infty}$$

$$= \lim_{x\to 1^-} \frac{x(\ln x)^2}{(1-x)} \to \frac{0}{0}$$

$$= \lim_{x\to 1^-} \frac{(\ln x)^2 + 2\ln x}{-1} = 0$$

$$\Rightarrow \lim_{x\to 1^-} (1-x)^{\ln x} = e^0 = 1$$

⑧ $\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} \to 1^\infty$

set $y = \frac{1}{x}\ln(1+x)$

$$\lim_{x\to 0^+} \frac{1}{x}\ln(1+x) \to 0\cdot\infty$$

$$= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \to \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = 1$$

$$\Rightarrow \lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e^1 = e$$

Exercises

1. $$\lim_{x\to 0^+} \frac{\tan^{-1}(2x)}{\ln x}$$

2. $$\lim_{x\to 0^+} \ln x - \ln(\sin x)$$

3 $$\lim_{x \to \infty} \frac{\log_2 x}{\log_3 (x+3)}$$

4 $$\lim_{x \to 0} \frac{e^{ax} - 1}{\sin x}$$

5 If $\lim_{x \to 0} \dfrac{\sin(ax) + bx^3 - 2x}{x^3} = 0$.

Find a and b.

Practice

1. $$\lim_{x\to 0^+} \sin x \ln x$$

2. $$\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$$

3. $$\lim_{x\to 0^+} x^{\sqrt{x}}$$

4. $$\lim_{x\to 0} \frac{3^x - 1}{2^x - 1}$$

5. $$\lim_{x\to \infty} e^x \left(1 - \cos\left(e^{-x}\right)\right)$$

6. $$\lim_{x\to 0} \frac{\sqrt{ax + a^2} - a}{x}$$

7. $$\lim_{x\to \infty} \frac{\ln\left(1 + e^x\right)}{1 + x}$$

8. $$\lim_{x\to \infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1}$$

9. $$\lim_{x\to \infty} x - \ln x$$

10. $$\lim_{x\to -\infty} \frac{x}{\sqrt{x^2+1}}$$

11. $$\lim_{x\to \frac{\pi}{4}} (1 - \tan x)\sec(2x)$$

12. $$\lim_{x\to \infty} \left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}}$$

13. $$\lim_{x\to 0} \frac{\sin^{-1} x}{x}$$