ITCS255
The Growth of Functions and Asymptotic Notations

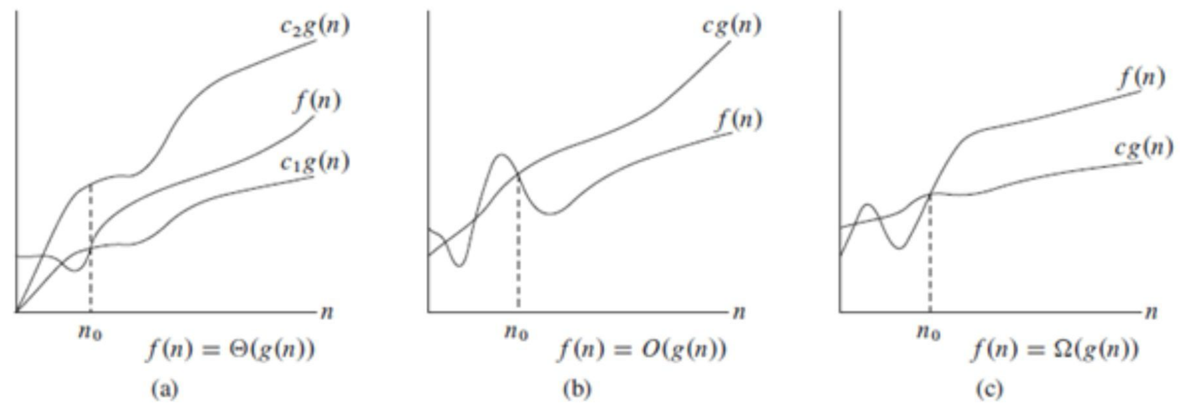
❖ Functions classification in order:

$$1 < logn < \sqrt{n} < n < nlog\, n < n^2 < n^3 < \cdots < 2^n < 3^n < \cdots < n^n$$

In which $logn$ is the smallest function and n^n is the greatest

❖ Types of asymptotic notations:

1- Big -O
2- Big -Ω
3- Big -Θ



In which:

1- Big -O means than it describes the upper bound

$$f(x) \in O\ g(x)$$

Where $g(x)$ is an upper bound of $f(x)$

2- Big -Ω means than it describes the lower bound

$$f(x) \in \Omega\ g(x)$$

Where $g(x)$ is a lower bound of $f(x)$

3- Big -Θ means than it describes the lower bound

$$f(x) \in \Omega\ g(x)$$

Where $g(x)$ is an upper bound and a lower bound of $f(x)$

❖ Starting with Big -O notation, in general is consists of 4 components:
1. $f(x)$ - function
2. $g(x)$ - function
3. Value C - constant
4. Value K - constant

Big -O way of representation is:

$$|f(n)| \leq C|g(x)| \text{ for all } n > k$$

Finding Big -O of a polynomial function can be in two ways:
1. Ad-hoc calculation
2. General procedure is simply taking the dominant term as g(x) and then having the summation of all terms to be as c.
   Note: General procedure can only be used on polynomials

Q1. Use the definition of Big -O to show that $x^4 + 9x^3 + 4x + 7$ is $O(x^4)$
1. Determine the dominant term which is x^4
2. Substitute all terms with the dominant term
$$x^4 + 9x^3 + 4x + 7 \leq x^4 + 9x^4 + 4x^4 + 7x^4$$
3. Simplify
$$\mathrm{x}^4 + 9\mathrm{x}^3 + 4\mathrm{x} + 7 \leq 21\mathrm{x}^4$$
4. Identify the needed parts g(x), c, k
   $g(x) = (x^4)$, $\mathrm{c} = 21$, $\mathrm{k} > 1$

Q2. Show that $\frac{x^3+2x}{2x+1}$ is $O(x^2)$

1. Substitute with the dominant term in the numerator and ignore all the terms after the dominant term in the denominator

$$\frac{x^3 + 2x}{2x + 1} \leq \frac{x^3 + 2x^3}{x}$$

2. Simplify and identify the needed parts g(x), c, k

$$\frac{x^3 + 2x}{2x + 1} \leq \frac{3x^3}{x}$$

$$\frac{x^3 + 2x}{2x + 1} \leq 3x^2$$

$g(x) = (x^2)\ , c = 3, k > 1$

Q3. Show that $\frac{x^2+1}{x+1}$ is $O(x)$

$$\frac{x^2 + 1}{x + 1} \leq \frac{x^2 + x^2}{x}$$

$$\frac{x^2 + 1}{x + 1} \leq \frac{2x^2}{x}$$

$$\frac{x^2 + 1}{x + 1} \leq 2x$$

$g(x) = x\ , c = 2, k > 1$

❖ Big -Ω which describes the lower bound.

Big - Ω way of representation is:

❖ $|f(n)| \geq C|g(x)|$ for all $n > k$

Big - Ω has two methods of solving the questions:

1. Ad-hoc calculation cases:
   - If all positive we simply take the dominant term $g(x)$= the function of the dominant term and c = the coefficient of the dominant term
   - Some negative there are some steps that will clarified in a question below
2. General procedure in which
   - $g(x)$ = the dominant term
   - d cases
     - all the numbers are positive k = 1 and c = the coffiecient of the dominant term
     - some negative numbers

       d = $\frac{2(\text{summation of all terms except the dominant})}{\text{the coffiecient of the dominant term}}$, k= max(1,d)

       and c = $\frac{\text{the coffiecient of the dominant term}}{2}$

Q1. Use basic definition to prove that $\sqrt{4n^2 - 3n + 2} \in \Omega(n)$

1. We first need to ignore all the positive terms and keep the negative terms

$$\sqrt{4n^2 - 3n + 2} \geq \sqrt{4n^2 - 3n}$$

2. Then we shall split the dominant term in to two terms $4n^2$ will be $3n^2$ and n^2

$$\sqrt{4n^2 - 3n + 2} \geq \sqrt{3n^2 + (n^2 - 3n)}$$

3. Simplify

$$\sqrt{4n^2 - 3n + 2} \geq \sqrt{3n^2 + n(n-3)}$$

If $(n-3) \geq 0$ then $\sqrt{3n^2 + n^2 - 3n} \geq \sqrt{3n^2}$ for $n \geq 3$

$$\sqrt{4n^2 - 3n + 2} \geq \sqrt{3}n$$

4. Identify the needed parts g(x), c, k

$$g(x) = n \, , c = \sqrt{3} \, , k = 3$$

Q2. Solve the following question using the general procedure
$3n^6 - 6n^5 + 4n^3 - 4n^2 - 7$

$$g(x) = n^6$$

And because there are some negative terms therefore, we need to find the value of d by

$$d = \frac{2(6 + 4 + 4 + 7)}{2} = 14$$

After finding the value of d we can find the value of k

$$k = \max(1,14) = 14$$

c= $\frac{3}{2}$

- Big -Θ is a combination of both Big -O and Big – Ω therefore whenever asked about Big -Θ we shall solve for both Big -O and Big – Ω.

Q1. Use the basic definition to prove that $\sqrt{9n^2 + n - 6} \in \theta(n)$

1. Finding Big-O

$$\sqrt{9n^2 + n - 6} \leq \sqrt{9n^2 + n^2 + 6n^2}$$

$$\leq \sqrt{16n^2}$$

$$\leq 4n$$

$$g(x) = n, c = 4, k = 1$$

2. Finding Big– Ω.

$$\sqrt{9n^2 + n - 6} \leq \sqrt{9n^2 - 6}$$

$$\leq \sqrt{8n^2 + (n^2 - 6n)}$$

$$\leq \sqrt{8n^2 + n(n - 6)}$$

If $(n - 6) \geq 0$ then $\sqrt{9n^2 + n - 6} \geq \sqrt{8n^2}$ for $n \geq 6$

$$\geq \sqrt{8}n$$

$$g(x) = n, c = \sqrt{8}, k = 6$$

❖ Using limits to prove bounds

In proving using limits we will be using some basic calculas derivatives rules.

Basic Derivatives Rules

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Some rules to keep in mind:

1. If $\frac{1}{\infty} = 0$ where $\frac{f(x)}{g(x)} = 0$ leads that $g(x)$ is a very big number and $f(x)$ is a small number in which $g(x)$ grows faster in this case and it is Big -O
2. If $\frac{\infty}{1} = \infty$ where $\frac{f(x)}{g(x)} = \infty$ leads that $f(x)$ is a very big number and $g(x)$ is a small number in which $f(x)$ grows faster in this case and it is Big - Ω
3. If it is finite "meaning the answer would be constant that leads on it being Big- θ

Q1. $f(x) = x^2 logx, g(x) = x^3$ determine which function grows faster

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{x^2 logx}{x^3} = \frac{logx}{x}$$

Using l'Hopital rule:

$$\lim_{x \to \infty} \frac{1}{x \ln(x)}$$

Note: $\ln(x)$ is considered as a constant

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Therefore $g(x)$ grows fatser.

Q2. $f(x) = 6x^3 - 5x^2 + 9$, $g(x) = x^3 logx$ determine which function grows faster

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\frac{6x^3-5x^2+9}{x^3logx}$$

$$\frac{6x^3}{x^3logx}-\frac{5x^2}{x^3logx}+\frac{9}{x^3logx}$$

$$\frac{6}{logx}-\frac{5}{xlogx}+\frac{9}{x^3logx}=0$$

Therefore $g(x)$ grows fatser.

Q3. Use infinite limits to prove that $\frac{4\mathrm{n}^3+2}{\mathrm{n}^2}\in\theta(\mathrm{n})$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\frac{\frac{4\mathrm{n}^3+2}{\mathrm{n}^2}}{n}$$

$$\lim_{x\to\infty}\frac{4\mathrm{n}^3+2}{n^3}$$

$$\lim_{x\to\infty}\frac{4\mathrm{n}^3}{n^3}+\frac{2}{n^3}$$

$$\lim_{x\to\infty}4+\frac{2}{n^3}\quad=4+0=4$$

Therefore it is Big- θ

❖ Combinations of Functions

Big -O rules:

1. $f_1(x)+f_2(x)\Longrightarrow O(f_1+f_2)\Longrightarrow O(\max\{g_{1,}g_2\}$

Meaning that the function that grows faster is going to be Big -O of the summation of 2 functions.

2. When having two different functions with different $g(x)$ we shall multiply both functions together.
3. When having the same $g(x)$ for two functions we shall write it once.

Q1. Give a big-Oestimate for $f(n) = (nlogn + n^2)(n^3 + 2)$

$$n^3 + 2 \in O\ (n^3)$$

$$nlogn + n^2 \ \in O\ (n^2)$$

$$(nlogn + n^2)(n^3 + 2) \in O(n^3 . n^2)$$

$$(nlogn + n^2)(n^3 + 2) \in O(n^5)$$